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# A TREATISE

ON

# HIGHER TRIGONOMETRY.

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ON

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BY THE

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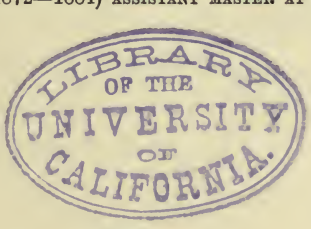


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# PREFACE.

THE present work is intended to complete the subject of Plane Trigonometry as far as it is usually read in Schools and in the junior Classes at the Universities. The introduction of the hyperbolic sine and cosine is an innovation which seems fully justified by their importance in other subjects, and by the simplification effected by their use in the statement of many theorems and formulæ. I wish to thank the Master of Gonville and Caius College for his valuable advice and assistance and for his permission to insert the proof given in Art. 62. The method of proof employed in Art. 47 was first suggested by Professor Adams.

I shall be very grateful for any suggestions or corrections from teachers or students.

J. B. L.

ETON,
*March*, 1884.

## NOTE.

References to the articles in the Higher Trigonometry are given thus [Art. 100]; references to the Elementary Trigonometry thus [E. 100].

The Articles and Exercises which are marked with a star should be omitted when the subject is read for the first time.

The order of the Chapters may in many cases be varied at the discretion of the teacher; in particular the last two Chapters may often be read as an Appendix to the Elementary Trigonometry.

Those of the examples which are not original, have been selected from the various Examination Papers which have been set at Cambridge in the Tripos and in the different College Examinations during the past forty years. Various Examination Papers are appended for the information of intending Candidates.

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# CHAPTER I.

## The Exponential and Logarithmic Series.

1. **The series**

$$1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\text{etc. ad inf.}$$

is of importance. Hence we prove as follows:

I. Its value is **less than 3.**

For it is less than

$$1+\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\text{etc.}$$

*i.e.* less than $1+1+\frac{1}{2}\left\{\frac{1}{1-\frac{1}{2}}\right\}$,

*i.e.* less than $1+1+1$.

II. Since it is less than 3, the series is **convergent.**

III. Its value is **2·71828182 ...**

This may be easily calculated [See Ex. I. (1)].

IV. It is **incommensurable.**

For suppose that it is commensurable; it can then be put into the form $\frac{m}{n}$ where $m$ and $n$ are integers. In this case

$$\frac{m}{n}=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\text{ etc. }+\frac{1}{\lfloor n}+\frac{1}{\lfloor n+1}+\text{ etc.}$$

Multiply each side of this supposed identity by $\underline{|n}$ then

$m\underline{|n-1}$ = a whole number $+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+$ etc.

But $$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\text{ etc.}$$

is a proper fraction; for it is greater than $\frac{1}{n+1}$ and less than

$$\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\text{ etc., } i.e. \text{ less than } \frac{1}{n}.$$

Hence we have to suppose that $m\underline{|n-1}$ (a whole number) = a whole number + a proper fraction; which is absurd.

V. Since the numerical value of the series is incommensurable, and we know of no surd or other algebraical expression that is equal to it, it is usual to express its numerical value by the letter $e$. [cf. E. 28.]

## EXAMPLES. I.

(1) Calculate the value of $e$ by taking the first 13 terms of the series.

(2) Prove that the first 13 terms of the series will give the value of $e$ correct to 9 significant figures.

(3) Prove that $\frac{2}{\underline{|1}}+\frac{4}{\underline{|3}}+\frac{6}{\underline{|5}}+\frac{8}{\underline{|7}}+\text{etc.}=e$.

(4) Prove that $\frac{1}{\underline{|2}}+\frac{1+2}{\underline{|3}}+\frac{1+2+3}{\underline{|4}}+\frac{1+2+3+4}{\underline{|5}}+\text{etc.}=\frac{e}{2}$.

(5) Prove that the series $1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+$ etc. is convergent for all values of $x$.

(6) Prove that the series $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+$etc. is convergent if $x$ is greater than $-1$ and is not greater than 1.

2. **Expansion of $e^x$ in ascending powers of $x$.**

Since $$\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x} \text{ always;}$$

and since by the binomial theorem

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\left(\frac{1}{n}\right) + \frac{nx\,(nx-1)}{1\,.\,2}\left(\frac{1}{n}\right)^{2} + \text{etc.}$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1\,.\,2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\,.\,2\,.\,3} + \text{etc.}$$

And similarly

$$\left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1\left(1-\frac{1}{n}\right)}{1\,.\,2} + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3} + \text{etc.}$$

Therefore

$$\left\{1 + 1 + \frac{1\left(1-\frac{1}{n}\right)}{1\,.\,2} + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3} + \text{etc.}\right\}^{x}$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1\,.\,2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\,.\,2\,.\,3} + \text{etc.}$$

This statement is arithmetically intelligible and true provided both these series are convergent.

They are convergent for all values of $n$ greater than 1.

Therefore they are arithmetically intelligible and true however great $n$ may be. And in the limit, when $n$ is infinitely increased, the above statement becomes [cf. Art. 8]

$$\left\{1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{etc.}\right\}^{x} = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc.}$$

or,
$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \text{etc.}$$

This result is called the **Exponential Theorem.**

3. To expand $a^x$ in ascending powers of $x$.

We have $$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc.}$$

Let $a$ be any number, and let $c = \log_e a$, so that $e^c = a$.

Then $$a^x = (e^c)^x = e^{cx} = e^{x \log_e a}.$$

Therefore

$$a^x = 1 + \frac{x \log_e a}{\lfloor 1} + \frac{x^2 (\log_e a)^2}{\lfloor 2} + \frac{x^3 (\log_e a)^3}{\lfloor 3} + \text{etc.}$$

## The Logarithmic Expansion.

4. In the above expansion put $1+y$ for $a$, and we obtain

$$(1+y)^x = 1 + \frac{x \log_e (1+y)}{\lfloor 1} + \frac{x^2 \{\log_e (1+y)\}^2}{\lfloor 2} + \text{etc.}$$

This may be put into a different form thus:

$$\frac{(1+y)^x - 1}{x} = \log_e (1+y) + \frac{x}{\lfloor 2} \{\log_e (1+y)\}^2 + \text{terms containing higher powers of } x$$

$$= \log_e (1+y) + x \,.\, R,$$

where $R$ is a quantity which is not infinite when $x = 0$.

The limit of the right-hand side when $x = 0$ is $\log_e (1+y)$.

The limit of the left-hand side may be found thus:

$$\frac{(1+y)^x - 1}{x} = \frac{1}{x}\left\{1 + xy + \frac{x(x-1)}{\lfloor 2} y^2 + \frac{x(x-1)(x-2)}{\lfloor 3} y^3 + \text{etc.} - 1\right\}$$

$$= y + \frac{x-1}{\lfloor 2} y^2 + \frac{(x-1)(x-2)}{\lfloor 3} y^3 + \text{etc.}$$

Now this, when $x = 0$, has for its limit

$$y - \tfrac{1}{2} y^2 + \tfrac{1}{3} y^3 - \tfrac{1}{4} y^4 + \text{etc.}$$

This series is convergent when $y$ is equal to or numerically less than 1.

Therefore, when $y$ lies between $-1$ and $+1$ or is equal to 1,

$$\log_e(1+y) = y - \tfrac{1}{2}\cdot y^2 + \tfrac{1}{3}\cdot y^3 - \tfrac{1}{4}\cdot y^4 + \text{ etc.}$$

This is the required **Logarithmic Expansion.**

## EXAMPLES. II.

(1) Calculate the numerical value of twelve terms of the series $1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\text{etc.}$ when $x=-1$, and show that it $=\frac{1}{e}$.

Prove the following statements:

(2) $\frac{2}{\lfloor 3}+\frac{4}{\lfloor 5}+\frac{6}{\lfloor 7}+\text{etc.}=\frac{1}{e}$.

(3) $\log_e 2=\frac{1}{2}+\frac{1}{3.4}+\frac{1}{5.6}+\text{etc.}=1-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-\text{etc.}$, and calculate the value of $\log_e 2$ to 2 decimal places.

(Result ·69...)

(4) $\frac{1}{y^2}+\frac{1}{2y^4}+\frac{1}{3y^6}+\text{etc.}=2\log_e y-\log_e(y+1)-\log_e(y-1)$.

(5) $\log_e\frac{1+y}{1-y}=2\{y+\frac{1}{3}y^3+\frac{1}{5}y^5+\text{etc.}\}$.

(6) $2\left\{\frac{1}{2y+1}+\frac{1}{3(2y+1)^3}+\frac{1}{5(2y+1)^5}+\text{etc.}\right\}$

$$=\log_e\frac{1+y}{y}=\log_e(1+y)-\log_e y.$$

(7) $2\log_e y-\log_e(y+1)-\log_e(y-1)$

$$=2\left\{\frac{1}{2y^2-1}+\frac{1}{3(2y^2-1)^3}+\frac{1}{5(2y^2-1)^5}+\text{etc.}\right\}.$$

(8) Prove that

$$\log_e(n+1)-\log_e(n-1)=2\left\{\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\text{etc.}\right\}.$$

(9) Use the series of Ex. (8) to prove that $\log_e 3=1{\cdot}098612$.

(10) Use the series of Ex. (7) and the result of Ex. (9) to prove that $\log_e 2=\cdot 693147$.

(11) Use the result of Ex. (9) and the series of Ex. (6) to prove that $\log_e 10=2{\cdot}302585$.

## On the Calculation of the Table of Logarithms.

5. The series for $\log_e(1+y)$ is only convergent provided $y$ is not greater than 1 and is greater than $-1$; also, unless $y$ is small, the series converges very slowly.

It is therefore not a convenient series for the purposes of numerical calculation.

We proceed to obtain such a series.

6. Since $\log_e(1+y)=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+$ etc. [Art. 4.]

therefore $\log_e(1-y)=-y-\frac{1}{2}y^2-\frac{1}{3}y^3-\frac{1}{4}y^4-$etc.

Hence by subtraction

$$\log_e\frac{1+y}{1-y}=2\left\{y+\tfrac{1}{3}y^3+\tfrac{1}{5}y^5+\text{ etc.}\right\},$$

where $y$ must not be numerically greater than 1.

Let $m$ and $n$ be positive integers, and let $m$ be $>n$; then $\frac{m-n}{m+n}$ is less than 1. Put $\frac{m-n}{m+n}$ for $y$ in the above result.

Then $$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \text{etc.}\right\}.$$

Let $m = n+1$ in the above; then

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \text{etc.}\right\}.$$

This series is rapidly convergent, and we have thus an easy method for obtaining the logarithms to the base $e$ of *successive* numbers.

Logarithms to the base $e$ are called **Napierian** Logarithms from their inventor. [E. 227.]

The logarithms are calculated thus:

Since $\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \text{etc.}\right\}$,

in this formula put 1 for $n$.

Then since $\log 1 = 0$, we can calculate the value of $\log_e 2$.

Next put 2 for $n$ in the above formula, and we can calculate the value of $\log_e 3$.

And so on.

The number of terms of the series which it is necessary to include diminishes as $n$ increases.

In this way a Table of the 'Napierian' Logarithms, of all whole numbers up to any desired magnitude may be determined.

## On the Calculation of Common Logarithms.

7. We know that $\log_{10} n = \log_e n \div \log_e 10$. [E. 209, 210.]

Hence, a table of Napierian Logarithms having been constructed, from it we take $\log_e 10$ (= 2·3025850...) and calculate $\frac{1}{\log_e 10} = \cdot 43429448$; the table of Common Logs of whole numbers is then formed by multiplying each of the corresponding Napierian Logarithms by ·43429448....

## EXAMPLES. III.

(1) From the preceding data calculate

$$\log_{10} 2, \quad \log_{10} 3, \quad \log_{10} 9.$$

(2) Find $\log_e 7$ and thence calculate $\log_{10} 7$.

(3) If $\mu = \log_{10} e$, prove that

$$\log_{10}(n+1) - \log_{10} n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.} \right\}.$$

8. In Art. 2 the limit, when $n$ is infinite, of

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\left(x-\frac{3}{n}\right)\ldots\left(x-\frac{r-1}{n}\right)}{\lfloor r},$$

is assumed to be $\frac{x^r}{\lfloor r}$.

This is clear as long as $r$ is not comparable with $n$.

That it is true for all values of $r$ may be proved by induction, thus:

Assume that the above expression $= \frac{x^r}{\lfloor r} + R$, where $R$ is a quantity whose limit is zero when $n$ is infinite; multiply each side by the factor $\left(x-\frac{r}{n}\right)\frac{1}{r+1}$, and we obtain

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\ldots\left(x-\frac{r}{n}\right)}{\lfloor r+1} = \frac{x^{r+1}}{\lfloor r+1} + \frac{x \,.\, R}{r+1} - \frac{1}{n}\cdot\frac{r}{r+1}\cdot\left\{\frac{x^r}{\lfloor r} + R\right\}.$$

In this, when $n = \infty$, $R = 0$, $\frac{1}{n} = 0$, and $\frac{r}{r+1}\frac{x^r}{\lfloor r}$ is finite however great $r$ may be. This proves the proposition.

9. *To prove that the limit of* $\left(\cos\frac{a}{n}\right)^n$, *when* n *is infinitely increased is* 1.

Since $\cos^2\frac{a}{n} = 1 - \sin^2\frac{a}{n}$;

$$\therefore \log\left(\cos\frac{a}{n}\right)^n = \log\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}} = \frac{n}{2}\log\left(1-\sin^2\frac{a}{n}\right)$$

$$= -\frac{n}{2}\left\{\sin^2\frac{a}{n} + \frac{1}{2}\sin^4\frac{a}{n} + \frac{1}{3}\sin^6\frac{a}{n} \text{ etc.}\right\}. \quad \text{[Art. 4.]}$$

[Art. 4, since $\sin^2\frac{a}{n}$ is less than 1.]

This series is less than the G. P.

$$\sin^2\frac{a}{n} + \sin^4\frac{a}{n} + \sin^6\frac{a}{n} + \text{etc.},$$

*i. e.* less than $\dfrac{\sin^2\frac{a}{n}}{1-\sin^2\frac{a}{n}}$, that is, than $\tan^2\frac{a}{n}$;

$$\therefore \log\left(\cos\frac{a}{n}\right)^n \text{ is less than } -\frac{1}{2}\left\{n\tan^2\frac{a}{n}\right\},$$

that is less than $-\frac{1}{2}\left\{\frac{n^2}{a^2}\tan^2\frac{a}{n}\right\}\frac{a^2}{n}$.

The limit of $\frac{n^2}{a^2}\tan^2\frac{a}{n}$ when $n=\infty$ is 1. [E. 290.]

The limit of $\frac{a^2}{n}$ when $n=\infty$ is 0.

$$\therefore \log\left(\cos\frac{a}{n}\right)^n = 0 \text{ when } n=\infty,$$

and therefore the limit of $\left(\cos\frac{a}{n}\right)^n$ is 1. Q. E. D.

10. *The limit of* $\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)^n$ *when* n *is infinitely increased is* 1.

We have (E. 289) 1, $\dfrac{\theta}{\sin\theta}$, $\dfrac{1}{\cos\theta}$ in ascending order of magnitude, when $\theta$ is less than 90°. Therefore also

$$(1)^n, \left(\frac{\frac{a}{n}}{\sin\frac{a}{n}}\right)^n, \frac{1}{\left(\cos\frac{a}{n}\right)^n},$$

are in ascending order of magnitude.

Now let $n$ be infinitely increased and then $\left(\dfrac{\frac{a}{n}}{\sin\frac{a}{n}}\right)^n$ lies between 1 and a quantity whose limit, by Art. 9, is 1. Q. E. D.

## *EXAMPLES. IV.

(1) Prove that the limit of $\left(\cos\frac{a}{n}\right)^r$ when $n$ is infinitely increased is 1.

(2) Prove that the limit of $\left(\frac{\sin\theta}{\theta}\right)^r$ when $\theta$ is infinitely diminished is 1.

(3) Prove that the limit of $\left(\cos\frac{a}{n}\right)^{n^2}$ is $e^{-\frac{a^2}{2}}$ when $n$ is infinitely increased.

(4) Prove that the limit of $\left(\cos\frac{a}{n}\right)^{n^3}$ is zero when $n$ is infinitely increased.

(5) Prove that the limit of $(\cos\theta)^{\frac{a^m}{\theta^m}}$ where $m$ is an integer, and $\theta$ is infinitely diminished is zero, $e^{-\frac{a^2}{2}}$, or 1 according as $m$ is greater, equal to, or less than 2.

## *MISCELLANEOUS EXAMPLES. V.

(1) Since $a^x = \{1+(a-1)\}^x$, prove by expanding the right-hand side that

$$a^x = 1 + A_1 x + A_2 x^2 + A_3 x^3 + \text{etc.},$$

where $A_1 = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3$.

(2) Since $a^{x+y} = a^x \times a^y$, expand $a^{x+y}$ and $a^x$ by the theorem of Ex. 1 and by equating coefficients of $x$, prove that

$$A_1 + 2A_2 y + 3A_3 y^2 + \text{etc.} = A_1 a^y.$$

Expand $a^y$, and by equating the coefficients of the various powers of $y$ find $A_2$, $A_3$, etc. in terms of $A_1$.

*Result.* $$a^x = 1 + A_1 x + \frac{A_1^2 x^2}{\lfloor 2} + \frac{A_1^3 x^3}{\lfloor 3} \text{ etc.}$$

(3) Show that $a^{\frac{1}{A_1}}$ in the last example is $e$.

Hence by Ex. 1 prove that

$$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{etc.}$$

(4) Prove that

$$\log_e n = m\{(n^{\frac{1}{m}} - 1) - \tfrac{1}{2}(n^{\frac{1}{m}} - 1)^2 + \tfrac{1}{3}(n^{\frac{1}{m}} - 1)^3 - \text{etc.}\}.$$

Hence, having given that $10^{\frac{1}{2^{32}}} - 1 = \cdot 000000000536112$, prove that $\log_e 10 = 2{\cdot}30258$.

(5) Prove that $\log(n+d) - \log n = \mu\left\{\frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \text{etc.}\right\}$.

Hence if $n$ be a number greater than 10000 and $d$ a number less than 1, prove that $\dfrac{\log(n+d) - \log n}{\log(n+d') - \log n} = \dfrac{d}{d'}$ to a sufficient approximation for all practical purposes.

(6) Prove that

$$\log\left(\frac{\sin\theta}{\theta}\right) = \log\cos\frac{\theta}{2} + \log\cos\frac{\theta}{2^2} + \log\cos\frac{\theta}{2^3} + \text{etc.}$$

(7) Prove that

$$\log\sin 2a + \log\cot a = \cos 2a - \tfrac{1}{2}\cos^2 2a + \tfrac{1}{3}\cos^3 2a - \text{etc.}$$

## CHAPTER II.

### De Moivre's Theorem.

11. Def. $\sqrt{-1}$ is a symbolical expression, whose square is $-1$, which is capable of obeying the ordinary laws of Algebra.

Since $\sqrt{-1}$ obeys the laws of Algebra

$$\sqrt{-a^2} = \sqrt{-1 \times a^2} = a\sqrt{-1}.$$

The student must observe that such an equation as $A + B\sqrt{-1} = a + b\sqrt{-1}$ can only be true when $A = a$ and $B = b$.

We shall often use the letter $i$ as an abbreviation for $\sqrt{-1}$.

12. **De Moivre's Theorem.** *Whatever be the value of* n *positive or negative, integral or fractional,* $\cos n\alpha + \sqrt{-1}\sin n\alpha$ *is* one *of the values of* $(\cos\alpha + \sqrt{-1}\sin\alpha)^n$.

I. When $n$ is a **positive integer.**

Consider the product

$$(\cos\alpha + \sqrt{-1}\sin\alpha) \times (\cos\beta + \sqrt{-1}\sin\beta).$$

It is equal to

$$\cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta + \sqrt{-1}\,(\cos\alpha \cdot \sin\beta + \sin\alpha \cdot \cos\beta).$$

That is to $\cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta)$.

Similarly the product

$$\{\cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta)\} \times \{\cos\gamma + \sqrt{-1}\sin\gamma\}$$

is equal to $\cos(\alpha+\beta+\gamma) + \sqrt{-1}\sin(\alpha+\beta+\gamma)$.

Proceeding in this way we obtain that the product of any number $n$ of factors, each of the form $\cos\alpha + \sqrt{-1}\sin\alpha$ is equal to

$$\cos(\alpha+\beta+\gamma+\ldots n \text{ terms}) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots n \text{ terms}).$$

In this result let $\beta = \gamma = \text{etc.} = \alpha$, and we have that

$$(\cos\alpha + \sqrt{-1}\sin\alpha)^n = \cos n\alpha + \sqrt{-1}\sin n\alpha.$$

Thus, when $n$ is a positive integer, De Moivre's Theorem is true.

II. When $n$ is a **negative integer.**

Let $n = -m$. Then $m$ is a positive integer. And

$$(\cos\alpha + \sqrt{-1}\sin\alpha)^n = (\cos\alpha + \sqrt{-1}\sin\alpha)^{-m}$$

$$= \frac{1}{(\cos\alpha + \sqrt{-1}\sin\alpha)^m} = \frac{1}{\cos m\alpha + \sqrt{-1}\sin m\alpha} \quad \text{[By I.]}$$

$$= \frac{1}{\cos m\alpha + \sqrt{-1}\sin m\alpha} \times \frac{\cos m\alpha - \sqrt{-1}\sin m\alpha}{\cos m\alpha - \sqrt{-1}\sin m\alpha}$$

$$= \frac{\cos m\alpha - \sqrt{-1}\sin m\alpha}{\cos^2 m\alpha + \sin^2 m\alpha}.$$

Therefore $(\cos\alpha + \sqrt{-1}\sin\alpha)^n = \cos m\alpha - \sqrt{-1}\sin m\alpha$

$= \cos(-m)\alpha + \sqrt{-1}\sin(-m)\alpha = \cos n\alpha + \sqrt{-1}\sin n\alpha.$

Thus De Moivre's Theorem is true when $n$ is a negative integer.

III. When $n$ is a **fraction**, positive or negative.

Let $n = \frac{p}{q}$, where $p$ and $q$ are integers.

Now $(\cos\beta + \sqrt{-1}\sin\beta)^q = \cos q\beta + \sqrt{-1}\sin q\beta$. [By I. and II.]

Therefore taking the $q^{\text{th}}$ root of both sides

$$\cos\beta + \sqrt{-1}\sin\beta$$

is one of the values of $(\cos q\beta + \sqrt{-1}\sin q\beta)^{\frac{1}{q}}$,

or, writing $\alpha$ for $q\beta$, $\cos\frac{\alpha}{q} + \sqrt{-1}\sin\frac{\alpha}{q}$

is one of the values of $(\cos\alpha + \sqrt{-1}\sin\alpha)^{\frac{1}{q}}$.

Therefore $\left(\cos\frac{\alpha}{q} + \sqrt{-1}\sin\frac{\alpha}{q}\right)^p$,

that is $\cos\frac{p\alpha}{q} + \sqrt{-1}\sin\frac{p\alpha}{q}$ [By I.]

is one of the values of $(\cos\alpha + \sqrt{-1}\sin\alpha)^{\frac{p}{q}}$.

Thus the theorem is completely established.

## EXAMPLES. VI.

(1) If $A$ stand for $\cos 2\alpha + i\sin 2\alpha$, and $B$, $C$, $D$ for similar expressions in terms of $\beta$, $\gamma$, $\delta$, prove that $AB + CD$
$= 2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta) + i\sin(\alpha+\beta+\gamma+\delta)\}$.

(2) With the notation of Ex. 1, prove that

$$\frac{1}{AB-CD} = \frac{\sin(\alpha+\beta+\gamma+\delta) - i\cos(\alpha+\beta+\gamma+\delta)}{2\sin(\alpha+\beta-\gamma-\delta)}.$$

(3) With the same notation, prove that $(A-B)(C-D)$
$= -4\sin(\alpha-\beta)\sin(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta) - i\sin(\alpha+\beta+\gamma+\delta)\}$.

(4) With the same notation prove that

$$\frac{1}{(A+B)(C+D)} = \frac{\cos(\alpha+\beta+\gamma+\delta) - i\sin(\alpha+\beta+\gamma+\delta)}{4\cos(\alpha-\beta)\cos(\gamma-\delta)}.$$

(5) Prove that $\cos(\alpha+\beta+\gamma\ldots)+i\sin(\alpha+\beta+\gamma+\ldots)$
$=\cos\alpha.\cos\beta.\cos\gamma\ldots\{(1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma)\ldots\ldots\}$.

(6) Prove that $\dfrac{\sin(\alpha+\beta+\gamma+\ldots)}{\cos\alpha.\cos\beta.\cos\gamma\ldots} = s_1 - s_3 + s_5 - s_4 + \text{etc.}$

where $s_1$ stands for the sum of $\tan\alpha+\tan\beta+\tan\gamma+$ etc., $s_3$ stands for the sum of the products of these tangents three at a time, and so on.

(7) Prove that $\tan(\alpha+\beta+\gamma+\ldots) = \dfrac{s_1 - s_3 + s_5 - \text{etc.}}{1 - s_2 + s_4 - \text{etc.}}$ where $s_1$, $s_2$,... are defined in Ex. 6.

(8) Write down the last term of the numerator of the fraction in Ex. 7, (i) when $n$ is even, (ii) when $n$ is odd.

13. It is known from the Theory of Equations that there are $q$ different values of $x$, and no more, which satisfy the equation $x^q = a$, where $a$ is real or of the form

$$A + \sqrt{(-1)}\,B.$$

We can prove that we may obtain $q$ different values by de Moivre's theorem, and no more.

14. The expression $\cos\theta + \sqrt{-1}\sin\theta$ is unaltered if for $\theta$ we put $(\theta + 2r\pi)$, where $r$ is an integer.

Hence $\cos\dfrac{\theta+2r\pi}{q} + \sqrt{-1}\sin\dfrac{\theta+2r\pi}{q}$,

which is one of the values of

$$\{\cos(\theta+2r\pi) + \sqrt{-1}\sin(\theta+2r\pi)\}^{\frac{1}{q}},$$

is one of the values of

$$\{\cos\theta + \sqrt{-1}\sin\theta\}^{\frac{1}{q}}.$$

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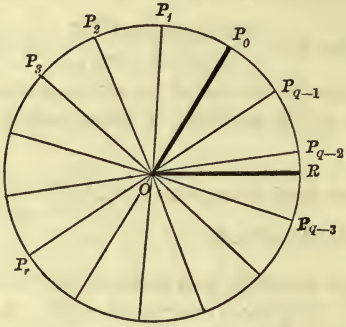
15. *By giving to* r *the values* 0, 1, 2 ... q − 1 *we obtain* q *different values of* $\cos\frac{\theta+2r\pi}{q}+\sqrt{-1}\sin\frac{\theta+2r\pi}{q}$; *and whatever integral value we give to* r, *we cannot obtain more than* q *different values.*




Take a circle, centre $O$ and radius $OR$. Let $ROP_0$ be the angle $\frac{\theta}{q}$. Divide the whole circumference of the circle starting from $P_0$, into $q$ *equal arcs*, $P_0P_1$, $P_1P_2$, $P_2P_3$, etc.

Then each of the angles $P_0OP_1$, $P_1OP_2$, $P_2OP_3$, ... etc. is equal to $\frac{2\pi}{q}$; and in describing the angle $\left(\frac{\theta}{q}+\frac{2r\pi}{q}\right)$, the revolving line, starting from $OR$, turns first into the position $OP_0$ and then on through $r$ of the angles $P_0OP_1$, $P_1OP_2$, etc. Hence, whatever integral value $r$ may have, $OP$ must stop in one of the $q$ positions $OP_0$, $OP_1$, $OP_2$, etc. and it can stop in no other position.

Therefore the expression $\cos\frac{\theta+2r\pi}{q}+\sqrt{-1}\sin\frac{\theta+2r\pi}{q}$ *cannot have* **more than** $q$ different values.

Also no two of these $q$ positions are equi-sinal and at the same time equi-cosinal.

Therefore this expression has $q$ *different* values.

Also by giving to $r$ the values 0, 1, 2, ... $(q-1)$, in succession, $OP$ will be made to stop in each of the $q$ possible positions in turn.

Therefore by giving to $r$ the values $0, 1, 2 \ldots (q-1)$ in succession, we obtain *the* q *different values* of the above expression. Q. E. D.

16. An expression of the form $A + \sqrt{-1}\,B$, where $A$ and $B$ are arithmetical quantities, can always be put into the form

$$r\{\cos\alpha + \sqrt{-1}\sin\alpha\}.$$

Let $A = r\cos\alpha$, $B = r\sin\alpha$. Then

$$A^2 + B^2 = r^2(\cos^2\alpha + \sin^2\alpha) = r^2, \text{ and } \frac{B}{A} = \frac{r\sin\alpha}{r\cos\alpha} = \tan\alpha;$$

whence $\alpha$ and $r$ can always be found. [E. 116.]

It will be convenient to take $r$ positive: then we must take $\alpha$ in that quadrant which makes $\cos\alpha$ the same sign as $A$. [Cf. E. 148, 149.]

EXAMPLE 1. *Express* $1+\sqrt{-1}$ *in the form* $r(\cos\alpha+\sqrt{-1}\sin\alpha)$.

Here $r\sin\alpha = 1$ and $r\cos\alpha = 1$, $\therefore r^2 = 2$, $\tan\alpha = 1$.

$$\therefore \quad 1+\sqrt{-1} = \sqrt{2}\{\cos 45^0 + \sqrt{-1}\sin 45^0\}.$$

Example 2. *Express* $(-a)$ *in the form* $r(\cos\alpha + i\sin\alpha)$.

Here $r\cos\alpha = -a$, $r\sin\alpha = 0$, $\therefore r^2 = a^2$, $\alpha = (2n+1)\pi$.

$\therefore -a = a\{\cos(2n+1)\pi + i\sin(2n+1)\pi\}$, where $n$ is an integer.

## EXAMPLES. VII.

(1) Express $1-\sqrt{-1}$, $\sqrt{3}+\sqrt{-1}$, $1+\sqrt{3}\sqrt{-1}$ each in the form $r(\cos\alpha+\sqrt{-1}\sin\alpha)$.

(2) Find all the values of (i) $(4\sqrt{2}+4\sqrt{2}\sqrt{-1})^{\frac{1}{3}}$, (ii) $(4\sqrt{3}+4\sqrt{-1})^{\frac{1}{3}}$, (iii) $(\sqrt{3}+\sqrt{-1})^{\frac{1}{5}}$.

(3) Find all the values of (i) $1^{\frac{1}{4}}$, (ii) $32^{\frac{1}{5}}$, (iii) $27^{\frac{1}{3}}$.

17. If we express any arithmetical quantity $a$ in the form of a De Moivre's expression we obtain

$$a\,(\cos 2r\pi + \sqrt{-1}\sin 2r\pi),$$

*i.e.* the product of $a$ by the De Moivre's expression for unity. Therefore the $n$ $n$th roots of any arithmetical quantity $a$ are found by multiplying the arithmetical $n$th root of $a$ by each of the $n$ $n$th roots of unity in succession.

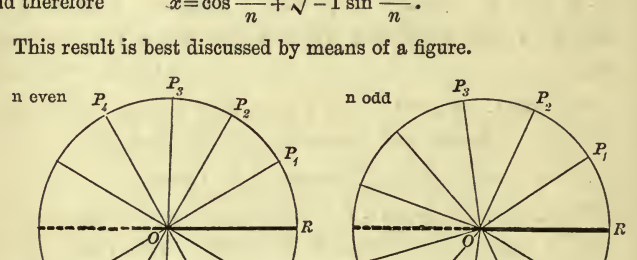
The $n$th roots of unity are therefore important, and are discussed in the following examples.

EXAMPLE 1. *Solve the equation* $x^n - 1 = 0$. In other words, *find all the values of* $\sqrt[n]{1}$, or, *find the factors of* $x^n - 1$.

Since $$\cos 2r\pi + \sqrt{-1}\sin 2r\pi = 1.$$

It follows that $x^n = \cos 2r\pi + \sqrt{-1}\sin 2r\pi$, where $r$ is an integer, and therefore $$x = \cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}.$$

This result is best discussed by means of a figure.




I. Let $n$ be a whole number.

Let the angle $ROP_1 = \frac{2\pi}{n}$. On the circumference of the circle centre $O$ and radius $OR$, measure off arcs $P_1P_2$, $P_2P_3$, etc. each equal to $RP_1$. Then since $n \cdot ROP_1 = 2\pi$, $n$ of these arcs will occupy the whole circumference, and $OP_n$ will coincide with $OR$. Also, if $r$ be a

whole number, in describing the angle $\frac{2r\pi}{n}$ the revolving line, starting from $OR$, must stop in one of the positions $OP_1$, $OP_2$, etc. and in no other. No two of these positions are both equi-sinal and equi-cosinal.

Thus the expression $\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}$

has $n$ different values, and no more; and these values can be found by giving $r$ in succession the values $0, 1, 2 \ldots n-1$.

When $r=0$, $x=1$: when $n$ is odd this is the only arithmetical value; when $n$ is even, there are two arithmetical values; for let $n=2m$, then when $r=m$, $x=-1$.

In any case, the angles $ROP_1$ and $ROP_{n-1}$ are equi-cosinal, and $\sin ROP_1 = -\sin ROP_{n-1}$. The same thing is true of $ROP_2$, $ROP_{n-2}$, and of $ROP_3$, $ROP_{n-3}$, and so on.

Hence $x-\cos\frac{2\pi}{n}-\sqrt{-1}\sin\frac{2\pi}{n}$, and $x-\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}$, are factors of $x^n-1$. Their product is

$$\left(x-\cos\frac{2\pi}{n}\right)^2+\sin^2\frac{2\pi}{n}, \text{ i.e. } x^2-2x\cos\frac{2\pi}{n}+1.$$

Hence we obtain that $m$ being a whole number

$x^{2m}-1=(x^2-1)\left(x^2-2x\cos\frac{2\pi}{2m}+1\right)\left(x^2-2x\cos\frac{4\pi}{2m}+1\right)\ldots m$ quadratic factors,

$x^{2m+1}-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{2m+1}+1\right)\left(x^2-2x\cos\frac{4\pi}{2m+1}+1\right)\ldots$

$m$ quadratic factors.

[Note. Let $\alpha=\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}$;

then $\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}=\alpha^r$.

Therefore the roots of the equation $x^n-1=0$ are $1, \alpha, \alpha^2, \alpha^3 \ldots \alpha^{n-1}$.]

II. When $n$ is a fraction in its lowest terms $=\frac{p}{q}$. Then $x^q-1=0$, or $x^p-1^q=0$, or $x^p-1=0$. This is the same as the case already discussed.

III. When $n$ is incommensurable (e.g. $\sqrt{2}$). Then as before

$$x=\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}.$$

In this case, $r$ being an integer and $n$ incommensurable, $\frac{2r\pi}{n}$ can never be an exact multiple of $2\pi$. The angles will therefore not recur geometrically and the equation will have one arithmetical root, viz. 1, and an unlimited number of symbolical roots.

## EXAMPLES. VIII.

(1) Find the roots of the equation $x^5-1=0$.

(2) Find the quadratic factors of $x^8-1$.

(3) Write down the quadratic factors of $x^{13}-1$.

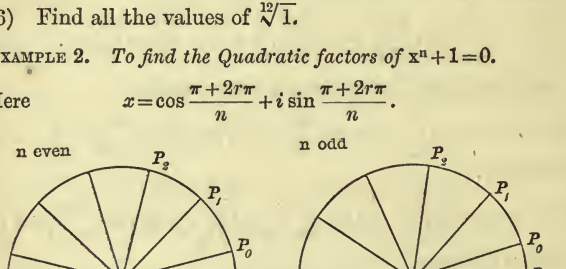
(4) Solve the equation $x^6-1=0$.

(5) Give the general quadratic factor of $x^{20}-a^{20}$.

(6) Find all the values of $\sqrt[12]{1}$.

EXAMPLE 2. *To find the Quadratic factors of* $x^n+1=0$.

Here $$x=\cos\frac{\pi+2r\pi}{n}+i\sin\frac{\pi+2r\pi}{n}.$$




In the figure $ROP_0=\frac{\pi}{n}$, $P_0OP_1=\frac{2\pi}{n}$, and $n$ angles each equal to $P_0OP_1$ make up $2\pi$; $OR$ bisects $P_0OP_{n-1}$. Also $ROP_0$ and $ROP_{n-1}$ are equi-cosinal, while $\sin ROP_0=-\sin ROP_{n-1}$, the same relation holds good for any two angles equi-distant from $OR$,

$\therefore \left(x - \cos\frac{\pi+2r\pi}{n} - i\sin\frac{\pi+2r\pi}{n}\right)$, and $\left(x - \cos\frac{\pi+2r\pi}{n} + i\sin\frac{\pi+2r\pi}{n}\right)$, are factors of $x^n+1$.

Therefore their product viz. $\left(x^2 - 2x\cos\frac{\pi+2r\pi}{n} + 1\right)$ is the form of the general quadratic factor of $x^n+1$.

When $n$ is even and $=2m$ there are $m$ such factors.

When $n$ is odd and $=2m+1$ there are $m$ such factors; the remaining factor is $x+1$, as is clear from the figure.

## EXAMPLES. IX.

(1) Find the roots of the equation $x^4+1=0$, and write down the quadratic factors of $x^4+1$.

(2) Write down the quadratic factors of $x^6+1$.

(3) Write down the general quadratic factor of $x^{20}+1=0$.

(4) Find all the values of $\sqrt[6]{-1}$. (5) Find the factors $x^{13}+1$.

(6) Find a general expression for all the values of $\sqrt[n]{-1}$.

## *MISCELLANEOUS EXAMPLES. X.

(1) Prove that

$$\sin(a_1+a_2+a_3 \ldots n \text{ terms}) = s_1c_{n-1} - s_3c_{n-3} + s_5c_{n-5} - \text{etc.},$$

where $s_rc_{n-r}$ stands for the sum of the products of the sines taken $r$ together each multiplied by the product of the remaining $n-r$ cosines.

(2) With the notation of Ex. 1, prove that

$$\cos(a_1+a_2+a_3 \ldots n \text{ terms}) = c_n - c_{n-2}s_2 + c_{n-4}s_4 - \text{etc.}$$

(3) Write down the expansion of

$\sin(a+\beta+\gamma+\delta+\epsilon)$ and of $\cos(a+\beta+\gamma+\delta+\epsilon)$.

(4) Prove that in the series of expressions formed by giving to $r$ in $\left(\cos\frac{\theta+2r\pi}{q}+i\sin\frac{\theta+2r\pi}{q}\right)$ the values 0, 1, 2, 3 ... $(q-1)$ in succession, the product of any two equidistant from the beginning and the end is constant.

(5) One value of $(\sqrt{3}+\sqrt{-1})^{\frac{15}{2}}$ is $-2^7(\sqrt{-1}+1)$.

(6) From the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

deduce by assuming $x=\cos 2\theta+i\sin 2\theta$, and corresponding assumptions for $a$, $b$ and $c$ that $\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\sin 2(\theta-a)+$ two similar expressions $=0$.

(7) Prove that the $n$ $n$th roots of unity form a series in G. P.

## CHAPTER III.

### Results of De Moivre's Theorem.

18. We proceed to deduce many important results from De Moivre's Theorem.

We shall generally in this chapter write $i$ for $\sqrt{-1}$.

19. By Art. 12, when $n$ is an integer we have

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n;$$

Expand the right-hand side of this identity by the binomial theorem, remembering that $i^2 = -1$ and that $i^4 = +1$. Equate the real part of the result to $\cos n\theta$. This gives us

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor 2} \cos^{n-2}\theta \,.\, \sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} \cos^{n-4}\theta \,.\, \sin^4\theta - \text{etc.}$$

Equate the imaginary part to $i \sin n\theta$. This gives us

$$\sin n\theta = n \cos^{n-1}\theta \,.\, \sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3} \cos^{n-3}\theta \,.\, \sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5} \cos^{n-5}\theta \,.\, \sin^5\theta - \text{etc.}$$

20. In the above $n$ is a positive integer, and the last terms in the series for $\cos n\theta$ and for $\sin n\theta$ will be different according as $n$ is even or odd.

## EXAMPLES. XI.

Prove the following statements :

(1) $\sin 4\theta = 4 \cos^3\theta \,.\, \sin\theta - 4 \cos\theta \,.\, \sin^3\theta.$

(2) $\cos 4\theta = \cos^4\theta - 6 \cos^2\theta \,.\, \sin^2\theta + \sin^4\theta.$

(3) The last term in the expansion of $\cos 10\theta$ is $-\sin^{10}\theta.$

(4) The last term in the expansion of $\sin 12\theta$ is $-12 \cos\theta \,.\, \sin^{11}\theta.$

(5) When $n$ is even the last term in the expansion of $\cos n\theta$ is $(-1)^{\frac{n}{2}} \sin^n\theta.$

(6) When $n$ is odd the last term in the expansion of $\cos n\theta$ is $(-1)^{\frac{n-1}{2}} n \cos\theta \,.\, \sin^{n-1}\theta.$

## Exponential Values of Sine and Cosine.

21. By De Moivre's Theorem, when $n$ is any commensurable number, and $x$ any angle,

$(\cos nx + i \sin nx)$ is a value of $(\cos x + i \sin x)^n$.

For $x$ put the unit of angular measurement; then

$(\cos n + i \sin n)$ is a value of $(\cos 1 + i \sin 1)^n$.

Let $k$ stand for $(\cos 1 + i \sin 1)$, then

$(\cos n + i \sin n)$ is a value of $k^n$,

where $k$ is independent of $n$.

Whatever other values $(\cos 1 + i \sin 1)^n$ may have, in what follows we shall only use the value $(\cos n + i \sin n)$.

22. This important result is a symbolical statement of the fact that **expressions of the form $\cos n + i \sin n$ are combined by the laws of indices.**

23. Let the unit of angle be a radian. [E. 59.]

Then since $\cos\theta + i \sin\theta = k^\theta$, [Art. 21.]

and consequently $\cos\theta - i \sin\theta = k^{-\theta}$,

where $k$ is independent of $\theta$,

$\therefore \quad 2i \sin\theta = k^\theta - k^{-\theta}$,

$$= 2\left\{\theta \log_e k + \frac{1}{\lfloor 3}\theta^3 (\log_e k)^3 + \text{etc.}\right\}. \quad \text{[Art. 3.]}$$

Hence $$i\frac{\sin\theta}{\theta} = \log_e k + \frac{1}{\lfloor 3}\theta^2 (\log_e k)^3 + \text{etc.}$$

$$= \log_e k + \theta^2 . R;$$

where $R$ is finite for all values of $\theta$ (since $\sin\theta$ is always less than $\theta$, and $\therefore \log_e k$ is finite).

Let $\theta$ be infinitely diminished. Then, since $\theta$ is the circular measure of the angle, the limit of $\frac{\sin\theta}{\theta}$ is 1. [E. 290.]

Also the limit of the right-hand side is $\log_e k$.

Hence $i = \log_e k$,

or, $k = e^{\sqrt{-1}}$.

Therefore, when $\theta$ is the circular measure of the angle,

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\sqrt{-1}\theta}.$$

24. Since $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$;

$\therefore$ $2\cos\theta = e^{i\theta} + e^{-i\theta}$; and $2i\sin\theta = e^{i\theta} - e^{-i\theta}$.

Hence $\dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

are **exponential values** of the cosine and sine respectively, *when the angle is expressed in circular measure.*

These results may be applied to prove any *general* formula in Elementary Trigonometry.

EXAMPLE. *Prove* $\dfrac{\sin 2a}{1+\cos 2a} = \tan a$.

$$\frac{2i\sin 2a}{2+2\cos 2a} = \frac{e^{2ia} - e^{-2ia}}{2 + e^{2ia} + e^{-2ia}} = \frac{(e^{ia}+e^{-ia})(e^{ia}-e^{-ia})}{(e^{ia}+e^{-ia})^2}$$

$$= \frac{e^{ia}-e^{-ia}}{e^{ia}+e^{-ia}} = \frac{2i\sin a}{2\cos a} = i\tan a. \quad \text{Q.E.D.}$$

## EXAMPLES. XII.

Use the exponential values of the sine and cosine to prove the following: (1) $\cos^2 a + \sin^2 a = 1$. (2) $\cos 2a = \cos^2 a - \sin^2 a$.

(3) $\sin\theta = -\sin(-\theta)$. (4) $\cos\theta = \cos(-\theta)$.

(5) $\cos(a+\beta).\cos(a-\beta) = \cos^2 a - \sin^2\beta = \cos^2\beta - \sin^2 a$.

(6) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (7) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(8) $2\cos na.\cos a = \cos(n+1)a + \cos(n-1)a$.

(9) $2\sin na\,(-1)\,2^2\sin^2 a$
$= 2\sin(n+2)a - 4\sin na + 2\sin(n-2)a$.

(10) $2\cos na\,(-1)\,2^2\sin^2 a$
$= 2\cos(n+2)a - 4\cos na + 2\cos(n-2)a$.

25. The results

$$2\cos\theta = e^{i\theta} + e^{-i\theta}, \quad 2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

may be used to simplify expressions containing $\sqrt{-1}$.

EXAMPLE 1. *Reduce* $\cos(\alpha+i\beta)$ *to the form* $A+iB$.

$$2\cos(\alpha+i\beta) = e^{i\alpha-\beta} + e^{-i\alpha+\beta} = e^{-\beta}\cdot e^{i\alpha} + e^{\beta}\cdot e^{-i\alpha}$$

$$= e^{-\beta}(\cos\alpha + i\sin\alpha) + e^{\beta}(\cos\alpha - i\sin\alpha)$$

$$= \cos\alpha\,(e^{\beta}+e^{-\beta}) - i\sin\alpha\,(e^{\beta}-e^{-\beta}).$$

This is in the required form.

EXAMPLE 2. *Express* $\log(a+ib)$ *in the form* $A+iB$.

Let $$a+ib = r(\cos\alpha + i\sin\alpha).$$

Then (Art. 16), $$r^2 = a^2+b^2, \quad \tan\alpha = \frac{b}{a}.$$

Thus, $$\log(a+ib) = \log\{r(\cos\alpha+i\sin\alpha)\} = \log r + \log e^{i\alpha}$$

$$= \log r + i\alpha = \tfrac{1}{2}\log(a^2+b^2) + i\tan^{-1}\frac{b}{a}.$$

This is in the required form.

EXAMPLE 3. *Reduce* $(a+ib)^{\alpha+i\beta}$ *to the form* $A+iB$.

Let $$a+ib = r(\cos\gamma + i\sin\gamma).$$

Then $$r^2 = a^2+b^2, \quad \tan\gamma = \frac{b}{a}.$$

And $$(a+ib)^{\alpha+i\beta} = r^{\alpha+i\beta}\cdot e^{i\gamma(\alpha+i\beta)}$$

$$= r^{\alpha}\cdot r^{i\beta}\cdot e^{i\gamma\alpha}\cdot e^{-\beta\gamma}$$

$$= r^{\alpha}\cdot e^{-\beta\gamma}\cdot e^{i\beta\log r}\cdot e^{i\gamma\alpha} \quad [\text{For } r = e^{\log r}]$$

$$= r^{\alpha}e^{-\beta\gamma}\cdot e^{i(\beta\log r+\alpha\gamma)}$$

$$= r^{\alpha}e^{-\beta\gamma}\{\cos(\beta\log r+\alpha\gamma) + i\sin(\beta\log r+\alpha\gamma)\}.$$

This is in the required form.

## EXAMPLES. XIII.

Prove the following statements :

(1) $\cos(a+i\beta)+i\sin(a+i\beta)=e^{-\beta}(\cos a+i\sin a)$.

(2) $2\sin(a+i\beta)=(e^{\beta}+e^{-\beta})\sin a+i(e^{\beta}-e^{-\beta})\cos a$.

(3) $\cos(a+i\beta)-i\sin(a+i\beta)=e^{\beta}(\cos a-i\sin a)$.

(4) $4\cos(a+i\beta).\cos(a-i\beta)=e^{2\beta}+e^{-2\beta}+2\cos 2a$.

(5) $4\sin(a+i\beta).\cos(a-i\beta)=2\sin 2a+i(e^{2\beta}-e^{-2\beta})$.

(6) $\sec(a+i\beta)=\dfrac{2(e^{\beta}+e^{-\beta})\cos a+2i(e^{\beta}-e^{-\beta})\sin a}{e^{2\beta}+2\cos 2a+e^{-2\beta}}$.

(7) $\tan(a+i\beta)=\dfrac{2\sin 2a+i(e^{2\beta}-e^{-2\beta})}{e^{2\beta}+2\cos 2a+e^{-2\beta}}$.

(8) $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}$.

(9) Express $a^{p+iq}$ in the form $A+iB$.

(10) Express $(a+ib)^i$ in the form $A+iB$.

(11) $\log\dfrac{a+ib}{a-ib}=2i\tan^{-1}\dfrac{b}{a}$.

(12) $\log\dfrac{\sin(x+iy)}{\sin(x-iy)}=2i\tan^{-1}\left\{\cot x\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}$.

(13) $\log\dfrac{\cos(x-iy)}{\cos(x+iy)}=2i\tan^{-1}\left\{\tan x\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}$.

(14) $\log\sin^2(x+iy)$
$=\log(e^{2y}-2\cos 2x+e^{-2y})-2i\tan^{-1}\left\{\cot x\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}$.

(15) $\log\cos^2(a+i\beta)$
$=\log(e^{2\beta}+2\cos 2a+e^{-2\beta})-2i\tan^{-1}\left\{\tan a\dfrac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}\right\}$.

(16) $a^{\sqrt{i}}=a^{\cos 45^0}\{\cos(\sin 45^0\log a)+i\sin(\sin 45^0\log a)\}$.

(17) Express $a^{\sqrt[3]{i}}$ in the form $A+iB$.

(18) Express $(a+ib+c^i)^{\alpha+i\beta}$ in the form $A+iB$.

26. Since

$$\cos\theta + i\sin\theta = e^{i\theta} = 1 + i\theta - \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \text{etc.},$$

we obtain by equating the real and imaginary parts

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \text{etc.},$$

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}$$

These results are very important.

In the next chapter will be found a proof independent of $\sqrt{-1}$ and a collection of examples.

## $e^{i\theta}$ IS A PERIODIC FUNCTION.

27. $\cos\theta$ and $\sin\theta$, and therefore also $\cos\theta + i\sin\theta$, repeat their values every time $\theta$ is increased by $2\pi$. Therefore $e^{i\theta}$ also repeats its values every time $\theta$ is increased by $2\pi$.

When a function of $\theta$ repeats every possible value in exactly the same order each time $\theta$ is increased by a certain value $\lambda$, it is said to be **periodic**, and $\lambda$ is called its **period.**

If we are given a particular value of such a function of $\theta$, we can find an unlimited number of values for $\theta$ (each of the form $\alpha + n\lambda$, where $n$ is a whole number,) for each of which this function will have that given value.

Also, as $\theta$ changes from 0 to $2\pi$, none of the values of $e^{i\theta}$ are repeated. In other words, there are no two values of $\theta$ *in the same period* for which $e^{i\theta}$ has equal values.

*Example.* Given that $\tan\theta = a$, and that $\alpha$ is one angle whose tangent is $a$, then we know that $\theta = \alpha + n\pi$, where $n$ is a whole number.

28. Hence, if $e^{i\theta} = e^{i\alpha}$, we know that $\theta$ and $\alpha$ differ by some multiple of $2\pi$, *i.e.* that $\theta = \alpha + 2n\pi$, where $n$ is a whole number, and the value of $n$ cannot be decided without some further datum.

*Example.* Since $2i \sin\theta = e^{i\theta} - e^{-i\theta}$, and $\sin\pi = 0$,

therefore $e^{i\pi} - e^{-i\pi} = 0$ or $e^{i\pi} = e^{-i\pi}$.

This means that $\pi$ and $-\pi$ are two values of $\theta$ for which the periodic function $e^{i\theta}$ has the same value. And since the period is $2\pi$, $\pi$ and $-\pi$ should differ by $2n\pi$. In this case $n$ is clearly 1.

29. The same thing may be stated thus:

since $\cos\alpha + i\sin\alpha = \cos(\alpha + 2r\pi) + i\sin(\alpha + 2r\pi)$,

$$\therefore\ e^{i\alpha} = e^{i\alpha + 2ir\pi} = e^{i\alpha} \cdot e^{2ir\pi}.$$

Therefore $e^{2ir\pi} = 1$ (as is also evident since

$$\cos 2r\pi + i\sin 2r\pi = 1).$$

Hence **unity has** one real logarithm, viz. 0, and also **an unlimited number of symbolical logarithms** each equal to $2ir\pi$, where $r$ is some integer.

30. Again, $a = a \times 1 = a \times e^{2ir\pi} = e^{\log_e a + 2ir\pi}$.

Hence **every real quantity** $a$ **has** one arithmetical logarithm, and also **an unlimited number of symbolical logarithms**, which differ by $2ir\pi$, where $r$ is an integer.

These symbolical logarithms do not interfere in any way with the theory of arithmetical logs explained in Chapter 1.

EXAMPLE. *Prove that the equation* $\sin\theta = 0$ *has no symbolical roots.*

Suppose that $\sin(\alpha + \sqrt{-1}\beta) = 0$ then

$$e^{i\alpha - \beta} - e^{-i\alpha + \beta} = 0, \text{ or } e^{2i\alpha - 2\beta} = 1 = e2ir\pi.$$

$\therefore\ 2i\alpha - 2\beta = 2ir\pi$; $\therefore\ \beta = 0$; which proves the proposition.

## EXAMPLES. XIV.

(1) Point out the fallacy in the following:

Since $2 \sin 3\pi = e^{i3\pi} - e^{-i3\pi}$ and $\sin 3\pi = 0$;

$$\therefore e^{i3\pi} = e^{-i3\pi}; \quad \therefore e^{6i\pi} = 1; \quad \therefore 6\pi = 0.$$

(2) Expose the fallacy in the following:

Let $a$ be any angle, then since

$$\cos(a-\pi) + i\sin(a-\pi) = \cos(a+\pi) + i\sin(a+\pi);$$

$$\therefore e^{i(a-\pi)} = e^{i(a+\pi)}; \quad \therefore a - \pi = a + \pi; \quad \therefore \pi = 0.$$

(3) Prove that the equation $\cos\theta = 0$ has no symbolical roots.

### TO EXPAND $\tan^{-1}x$ IN TERMS OF $x$.

31. Since $2\cos a = e^{ia} + e^{-ia}$, and $2i \sin a = e^{ia} - e^{-ia}$,

therefore $$i \tan a = \frac{2i \sin a}{2 \cos a} = \frac{e^{ia} - e^{-ia}}{e^{ia} + e^{-ia}}; \qquad \text{[Art. 24.]}$$

$$\therefore \frac{1 + i\tan a}{1 - i\tan a} = \frac{2e^{ia}}{2e^{-ia}} = e^{2ia};$$

$$\therefore \log \frac{1 + i\tan a}{1 - i\tan a} = \log e^{2ia} = 2ia + 2in\pi. \qquad \text{[Art. 28.]}$$

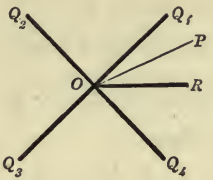
Hence, expanding the left-hand side by Art. 5, we have

$$2ia + 2in\pi = 2\{i\tan a + \tfrac{1}{3}(i\tan a)^3 + \tfrac{1}{5}(i\tan a)^5 + \text{etc.}\}$$

or $$a + n\pi = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \text{etc.}$$

This series is convergent if $\tan a$ is equal to or less than

unity. It is therefore arithmetically intelligible and true, provided $a$ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, or between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$,




and so on [*i.e.* provided $OP$ stops within the right angle $Q_4OQ_1$ or within the right angle $Q_2OQ_3$ in the figure].

32. When $a$ lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$, the value of $n$ is 0. For $n$ is an *integer* (or zero), while the value of $\pi$ is known to be greater than 3 [E. 37], and as $a$ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$ the numerical value of the series

$$\tan a - (\tfrac{1}{3}\tan^3 a - \tfrac{1}{5}\tan^5 a) - (\tfrac{1}{7}\tan^7 a - \tfrac{1}{9}\tan^9 a) - \ldots$$

is always less than $\tan a$, and therefore less than 1.

Hence, when $a$ lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$, we have

$$a = \tan a - \tfrac{1}{3}\tan^3 a + \tfrac{1}{5}\tan^5 a - \ldots$$

This result is called **Gregory's Series.**

Similarly we can prove that when $a$ lies between $\frac{3}{4}\pi$ and $\frac{5}{4}\pi$, $\quad a - \pi = \tan a - \frac{1}{3}\tan^3 a + \frac{1}{5}\tan^5 a \ldots$
and so on.

33. In this result put $a = \frac{\pi}{4}$, and we obtain

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{etc.}$$

34. The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$ is very slowly convergent; we shall therefore show how series which are more rapidly convergent may be obtained from Gregory's Series.

35. **Euler's Series.**

Since $\tan^{-1}\frac{1}{2}+\tan^{-1}1=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$.

Let $\alpha=\tan^{-1}\frac{1}{2}$, or $\tan\alpha=\frac{1}{2}$,

and $\alpha=\tan\alpha-\frac{1}{3}\tan^3\alpha+\frac{1}{5}\tan^5\alpha-\text{etc.},$ [Art. 31.]

$$\therefore\ \alpha=\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}-\text{etc.}$$

Let $\beta=\tan^{-1}\frac{1}{3}$, then

$$\beta=\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\text{etc.},$$ [Art. 31.]

and $\frac{1}{4}\pi=\alpha+\beta=$ the sum of these two series.

36. **Machin's Series.**

Since $2\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{5}{12}$ and $2\tan^{-1}\frac{5}{12}=\tan^{-1}\frac{120}{119}$,

$$\therefore\ 4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239}=\tfrac{1}{4}\pi.$$

Hence
$$\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\text{etc.}\right\}$$
$$-\left\{\frac{1}{239}-\frac{1}{3}\cdot\frac{1}{239^3}+\frac{1}{5}\cdot\frac{1}{239^5}-\text{etc.}\right\}.$$ [Art. 31.]

## EXAMPLES. XV.

(1) Prove that $\frac{\pi}{6}=\frac{1}{\sqrt{3}}\left\{1-\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{5}\cdot\frac{1}{3^2}-\frac{1}{7}\cdot\frac{1}{3^3}+\text{etc.}\right\}$.

(2) Prove that $4\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{120}{119}$. Hence prove that if $\frac{1}{4}\pi=4\tan^{-1}\frac{1}{5}-x$, then $x=\frac{1}{239}$.

(3) Prove that $\frac{\pi}{8}=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\text{etc.}$; hence calculate the value of $\pi$ to 2 decimal places.

(4) Calculate the value of $\pi$ to 3 decimal places by the aid of Euler's series.

(5) Calculate the value of $\pi$ to 3 decimal places by the aid of Machin's series.

(6) Prove that $\frac{1}{4}\pi=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$.

37. Let $\cos\theta + i\sin\theta = x,$

then $\cos\theta - i\sin\theta = \frac{1}{x} = x^{-1},$ [Art. 12.]

and $2\cos\theta = x + x^{-1},\ 2i\sin\theta = x - x^{-1}.$

[It should be observed that in the equation $2\cos\theta = x + x^{-1}$, either $x$ or $\theta$ must be symbolical. For if $\theta$ be real $2\cos\theta$ is less than 2. If $x$ be real $x + x^{-1}$ is numerically greater than 2. $x$ of course stands for $e^{i\theta}$.]

Also, $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$ [Art. 12.]

$$x^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$$

$$\therefore\ 2\cos n\theta = x^n + x^{-n},\ \text{and}\ 2i\sin n\theta = x^n - x^{-n}.$$

Hence $(2\cos\theta)^n = (x + x^{-1})^n$

$$= x^n + n\,.\,x^{n-2} + \frac{n(n-1)}{\lfloor 2}\,x^{n-4} + \text{etc.} + nx^{-(n-2)} + x^{-n}$$

$$= (x^n + x^{-n}) + n\,(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{\lfloor 2}\,(x^{n-4} + x^{-(n-4)}) + \text{etc.}$$

$$= 2\cos n\theta + n\,.\,2\cos(n-2)\,\theta + \frac{n\,(n-1)}{\lfloor 2}\,2\cos(n-4)\,\theta + \text{etc.}$$

Also $(2i\sin\theta)^n = (x - x^{-1})^n.$

First let $n$ be **even.** Then the expansion of $(x - x^{-1})^n$ is

$$x^n + x^{-n} - n\,(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{\lfloor 2}\,(x^{n-4} + x^{-(n-4)}) - \text{etc.}$$

$$\therefore\ (2i\sin\theta)^n = 2\cos n\theta - n\,2\cos(n-2)\,\theta + \frac{n(n-1)}{\lfloor 2}\,2\cos(n-4)\,\theta - \text{etc.}$$

Next let $n$ be **odd.** Then the expansion of $(x - x^{-1})^n$ is

$$x^n - x^{-n} - n\,(x^{n-2} - x^{-(n-2)}) + \frac{n(n-1)}{\lfloor 2}\,(x^{n-4} - x^{-(n-4)}) - \text{etc.}$$

$$\therefore\ (2i\sin\theta)^n = 2i\sin n\theta - n\,.\,2i\sin(n-2)\,\theta + \frac{n(n-1)}{\lfloor 2}\,2i\sin(n-4)\,\theta - \text{etc.}$$

Whence dividing by $i$ and putting $i^2 = -1$ we have

$$(-1)^{\frac{n-1}{2}} 2^n \sin^n\theta = 2\sin n\theta - n \,.\, 2\sin(n-2)\theta - \text{etc.}$$

It must be noticed that when the last term is independent of $\theta$, the factor 2 is omitted.

## EXAMPLES. XVI.

Prove that

(1) $128\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35.$

(2) $64\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$

(3) $64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$

(4) $512\sin^{10}\theta = \cos 10\theta - 10\cos 8\theta + 45\cos 6\theta - 120\cos 4\theta + 210\cos 2\theta - 126.$

38. *To resolve* $x^{2n} - 2x^n \cos n\theta + 1$ *into factors, when* n *is a whole number.*

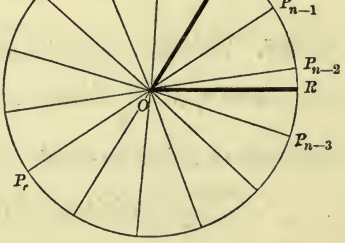
Since $x^{2n} - 2x^n\cos n\theta + 1 = (x^n - \cos n\theta)^2 + \sin^2 n\theta$

$= \{(x^n - \cos n\theta) + i\sin n\theta\}\{(x^n - \cos n\theta) - i\sin n\theta\}$

$= \{x^n - (\cos n\theta - i\sin n\theta)\}\{x^n - (\cos n\theta + i\sin n\theta)\}$

$= \{x^n - (\cos\theta - i\sin\theta)^n\}\{x^n - (\cos\theta + i\sin\theta)^n\},$

therefore $x - (\cos\theta - i\sin\theta)$ and $x - (\cos\theta + i\sin\theta)$ are factors of $x^{2n} - 2x^n\cos n\theta + 1$.



Therefore also their product, *i.e.* $x^2-2x\cos\theta+1$, is a factor of $x^{2n}-2x^n\cos n\theta+1$.

And since $\cos n\theta$ is unaltered if for $\theta$ we write $\theta+\dfrac{2r\pi}{n}$ it follows that $x^2-2x\cos\left(\theta+\dfrac{2r\pi}{n}\right)+1$ is also a factor.

In the above figure let $ROP_0=\theta$, and let the whole circumference, starting from $P_0$, be divided into $n$ equal arcs $P_0P_1$, $P_1P_2$ ... $P_{n-1}P_0$. Hence, whatever be the integral value of $r$, the angle $\theta+\dfrac{2r\pi}{n}$ is represented by one of the angles $ROP_0$, $ROP_1$, etc.

Hence *in general* there are $n$ different values and no more of $\cos\left(\theta+\dfrac{2r\pi}{n}\right)$.

[The exceptions are (i) when one of the points $P_r$ coincides with $R$, (ii) when $R$ *bisects* one of the arcs $P_rP_{r-1}$; *i.e.* (i) when $n\theta=2r\pi$, (ii) when $n\theta=(2r+1)\pi$, and in these cases $x^{2n}-2x^n\cos n\theta+1$ reduces (i) to the form $(x^n-1)^2=0$, (ii) to the form $(x^n+1)^2=0$; the factors of these forms have been discussed on pp. 18, 19.]

And the $n$ different values are found by giving to $r$ the values 0, 1, 2 ... $(n-1)$ in succession.

Hence the $n$ quadratic factors of $x^{2n}-2x^n\cos n\theta+1$ are

$$(x^2-2x\cos\theta+1)\left\{x^2-2x\cos\left(\theta+\frac{2\pi}{n}\right)+1\right\}\times\ldots$$

$$\times\left\{x^2-2x\cos\left(\theta+\frac{n-1}{n}2\pi\right)+1\right\}.$$

## EXAMPLES. XVII.

Solve the following equations:

(1) $x^8-2x^4\cos 60^0+1=0$. (2) $x^{10}-2x^5\cos 10^0+1=0$.

(3) $x^{12}-2x^6\cos\frac{2}{3}\pi+1=0$. (4) $x^{10}+\sqrt{3}x^5+1=0$.

(5) Write down the factors of $x^{2n}-2x^ny^n\cos a+y^{2n}$.

## *MISCELLANEOUS EXAMPLES. XVIII.

(1) Prove that if $\cos\theta$ and $\sin\theta$ be defined by the equations $2\cos\theta = a^{\theta} + a^{-\theta}$, $2i\sin\theta = a^{\theta} - a^{-\theta}$, then $\sin\theta$ and $\cos\theta$ satisfy the fundamental conditions

$$\cos^2\theta + \sin^2\theta = 1, \quad \cos\theta = \cos(-\theta), \quad \sin\theta = -\sin(-\theta).$$

(2) Prove that if a degree is the unit of angular measurement $2\cos x = k^x + k^{-x}$ where $k^{180} = e^{i\pi}$.

(3) Assuming De Moivre's theorem, prove that

$$\frac{i\sin x^0 - (1-\cos x^0)}{x} = \log k + \frac{x}{\underline{2}}(\log k)^2 + \text{etc.},$$

where $k = \cos 1^0 + i\sin 1^0 = e^{\frac{i\pi}{180}}$.

(4) Prove that if two right angles be taken as the unit of angle, the exponential values of $\cos x$ and $\sin x$ are $\frac{1}{2}(e^{i\pi x} + e^{-i\pi x})$ and $-\frac{1}{2}i(e^{i\pi x} - e^{-i\pi x})$.

(5) Assuming that $e^{ir\alpha} = 1$ where $r$ is an integer, prove that $e^{i\theta}$ is a periodic function of $\theta$.

(6) Assuming that $e^{i\theta}$ is a periodic function of $\theta$, and that the period is $2\pi$, prove that $e^{2ir\pi} = 1$ where $r$ is an integer.

(7) Prove that $(ib + c^i)^a = r^a(\cos a\theta + i\sin a\theta)$ where

$$r^2 = 1 + b^2 + 2b\sin\log c, \quad \tan\theta = \frac{b + \sin\log c}{\cos\log c}.$$

(8) If $\log(1 + \cos 2\theta + i\sin 2\theta) = A + iB$, then

$$A = \log 2 + \log(\cos\theta).$$

(9) Prove that

$$e^x - 2\cos\theta + e^{-x} = 4\sin\tfrac{1}{2}(\theta + ix)\sin\tfrac{1}{2}(\theta - ix).$$

(10) If $\cos^{-1}(\alpha + i\beta) = A + iB$, prove that

$$\frac{a^2}{\cos^2 A} - \frac{\beta^2}{\sin^2 A} = 1, \text{ and } \frac{a^2}{(e^{\beta} + e^{-\beta})^2} + \frac{\beta^2}{(e^{\beta} - e^{-\beta})^2} = \tfrac{1}{4}.$$

(11) Prove that $\log_e(-1)=i(2n+1)\pi$.

(12) Prove that

$$\log(x+iy)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}.$$

Hence prove that

$$\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta}=x\sin\theta-\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\ldots\ldots$$

(13) If $\phi(a)$ is such a function of $a$ that

$$\phi(a)\times\phi(\beta)=\phi(a+\beta)$$

for all values of $a$ and $\beta$, prove that $\phi(a)=\{\phi(1)\}^a$ for all rational values of $a$. Show that $\cos a+i\sin a$ is a form of $\phi(a)$ which satisfies the preceding equation, and deduce De Moivre's theorem.

(14) Prove that

$$\tan^{-1}(\cos a+i\sin a)=(n+\tfrac{1}{4})\pi+i\log(\tan\tfrac{1}{2}a).$$

(15) Prove that

$$\frac{\pi}{4}=4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{70}+\tan^{-1}\tfrac{1}{99},$$

and apply the result to find the value of $\pi$ to 5 places of decimals.

(16) Find the number of radians in the least angle whose tangent is $\frac{1}{10}$; also the number of degrees in the least angle whose tangent is 10.

(17) Prove that the general value of $e^{i\theta}$ is

$$\cos(1+2r\pi)\theta+i\sin(1+2r\pi)\theta$$

where $r$ is an integer.

(18) *Defining* $\cos\theta$ as the real part and $i\sin\theta$ as the imaginary part of $e^{i\theta}$, prove

$$\cos\theta=\cos(-\theta),\ \sin\theta=-\sin(-\theta),\ \sin(\theta+\phi)$$
$$=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

## CHAPTER IV.

### Proofs without the use of $\sqrt{-1}$.

39. In this Chapter we shall give proofs of most of the preceding results by methods which do not involve the use of $\sqrt{-1}$.

The student must not on this account suppose that the validity of results obtained by the aid of $\sqrt{-1}$ is doubtful. We shall make some remarks on this point later on.

40. *To prove, when* n *is a positive integer,*

$$\cos n\alpha = \cos^n \alpha - \frac{n(n-1)}{1.2} \cos^{n-2}\alpha . \sin^2\alpha + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \cos^{n-4}\alpha . \sin^4\alpha - \text{etc.},$$

*and*

$$\sin n\alpha = n \cos^{n-1}\alpha . \sin\alpha - \frac{n(n-1)(n-2)}{1.2.3} \cos^{n-3}\alpha . \sin^3\alpha + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \cos^{n-5}\alpha . \sin^5\alpha - \text{etc.}$$

These formulæ may be proved by **induction**, thus: Assuming that the above statements are true for a certain value of $n$, we can prove that they must also be true when $n+1$ is written for $n$.

Since $\cos(n+1)\alpha = \cos n\alpha \,.\, \cos\alpha - \sin n\alpha \,.\, \sin\alpha$ [E. 154.]; in this, substitute for $\cos n\alpha$ and $\sin n\alpha$ those values given above, and we obtain

$$\cos(n+1)\alpha = \cos^{n+1}\alpha - \left\{\frac{n(n-1)}{1\,.\,2} + n\right\}\cos^{n-1}\alpha \,.\, \sin^2\alpha$$

$$+ \left\{\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\right\}\cos^{n-3} \,.\, \sin^4\alpha - \text{etc.}$$

The coefficient of $\cos^{n-r}\alpha \,.\, \sin^{r+1}\alpha$ is

$$(-1)^{\frac{r+1}{2}}\left\{\frac{n(n-1)\ldots(n-r+1)(n-r)}{\lfloor r+1} + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}\right\}$$

$$= (-1)^{\frac{r+1}{2}}\left\{\frac{(n+1)\,n(n-1)\ldots(n-r+1)}{\lfloor r+1}\right\}.$$

Therefore $\cos(n+1)\alpha = \cos^{n+1}\alpha - \dfrac{(n+1)\,.\,n}{1\,.\,2}\cos^{n-1}\alpha \,.\, \sin^2\alpha$

$$+ \frac{(n+1)\,n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}\cos^{n-3}\alpha \sin^4\alpha - \text{etc.}$$

A similar result will hold good for $\sin(n+1)\alpha$.

Thus, if the formulæ are true when $n$ is any whole number, they are true when $n+1$ is substituted for $n$;

But they are true when $n=1$ and when $n=2$;

Therefore they are true when $n$ is any whole number.

41. *To prove* $\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \text{etc.},$

*and* $\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} + \text{etc.}$

In the formula

$$\cos n\alpha = \cos^n\alpha - \frac{n(n-1)}{1.2}\cos^{n-2}\alpha \,.\, \sin^2\alpha + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\alpha \,.\, \sin^4\alpha + \text{etc.}$$

write $\theta$ for $n\alpha$, and let $n$ be increased without limit while $\theta$ remains unchanged. Then since $\alpha = \frac{\theta}{n}$, $\alpha$ must be diminished without limit. We may write the above in the following form,

$$\cos\theta = \left(\cos\frac{\theta}{n}\right)^n - \frac{\theta(\theta-\alpha)}{1.2}\left(\cos\frac{\theta}{n}\right)^{n-2}\left(\frac{\sin\alpha}{\alpha}\right)^2 + \frac{\theta(\theta-\alpha)(\theta-2\alpha)(\theta-3\alpha)}{1.2.3.4}\left(\cos\frac{\theta}{n}\right)^{n-4}\left(\frac{\sin\alpha}{\alpha}\right)^4 - \text{etc.}$$

When $n$ is infinitely increased, the limit of $\left(\cos\frac{\theta}{n}\right)^r$ is 1, since $r$ is not greater than $n$; [Art. 9.]

the limit of $\frac{\theta(\theta-\alpha)(\theta-2\alpha)\ldots(\theta-\overline{r-1}\alpha)}{\lfloor r}$ is $\frac{\theta^r}{\lfloor r}$ [Art. 8];

and the limit of $\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^r$ is 1, since $r$ is not greater than $n$.

[Art 10.]

Therefore, by proceeding to the limit, we obtain

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} + \text{etc.}$$

Similarly, the expansion for $\sin na$ may be written

$$\sin\theta = \theta \cdot \left(\cos\frac{\theta}{n}\right)^{n-1}\left(\frac{\sin a}{a}\right) - \frac{\theta(\theta-a)(\theta-2a)}{1.2.3}\left(\cos\frac{\theta}{n}\right)^{n-3}\left(\frac{\sin a}{a}\right) + \text{etc.}$$

By proceeding to the limit, we obtain as before

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}$$

42. In the result $\sin a = a - \frac{a^3}{\lfloor 3} + \frac{a^5}{\lfloor 5} - \frac{a^7}{\lfloor 7} + \text{etc.}$ the series is convergent for *all* values of $a$.

[For the ratio of any term to the preceding is $\frac{a^2}{n(n+1)}$; and whatever be the value of $a$, by taking $n$ large enough this fraction can be made less than some quantity which is itself less than unity.]

In the proof of Art. 41 no limit was put upon the value of the angle. Therefore the result is arithmetically intelligible and true for all values of $a$.

Therefore the series $a - \frac{a^3}{\lfloor 3} + \frac{a^5}{\lfloor 5} - \frac{a^7}{\lfloor 7} + \text{etc.}$, which is equal to $\sin a$ for all values of $a$, must be **periodic.** [Art. 27.]

43. A series in *ascending* powers of a quantity ($a$) is chiefly useful when $a$ is small, for the smaller the quantity $a$, the greater is the relative importance of the *earlier* terms of the series. Also, the sine (and the cosine) of an angle of any magnitude may be expressed in terms of the sine or cosine of some angle less than $\frac{1}{4}\pi$.

Hence, the above series are never used in numerical calculations except for values of $a$ less than $\frac{1}{4}\pi$.

44. We have

$$\sin a = a - \left(\frac{a^3}{\lfloor 3} - \frac{a^5}{\lfloor 5}\right) - \left(\frac{a^7}{\lfloor 7} - \frac{a^9}{\lfloor 9}\right) - \text{etc.}$$

$$= \left(a - \frac{a^3}{\lfloor 3}\right) + \left(\frac{a^5}{\lfloor 5} - \frac{a^7}{\lfloor 7}\right) + \left(\frac{a^9}{\lfloor 9} - \frac{a^{11}}{\lfloor 11}\right) + \text{etc.}$$

Each of the above brackets is *positive* (provided $a^2$ is not greater than 6 and therefore, *a fortiori*, if $a$ is less than 1). Therefore $\sin a$ is less than $a$ and greater than $a - \frac{1}{6}a^3$.

Again

$(\sin a - a)$ is *negative* and $= -\frac{1}{6}a^3 +$ (a positive quantity),

$\sin a - (a - \frac{1}{6}a^3)$ is *positive* and $= \frac{1}{120}a^5 -$ (a positive quantity).

Therefore the difference between $a$ and $\sin a$ is less than $\frac{1}{6}a^3$; the difference between $\sin a$ and $a - \frac{1}{6}a^3$ is less than $\frac{1}{120}a^5$.

EXAMPLE. If $a = \frac{1}{10}$ (of a radian), the difference between $a$ and $\sin a$ is less than $\frac{1}{6} . 10^{-3}$, *i.e.* less than a six-hundredth part of $a$. The difference between $\sin a$ and $a - \frac{1}{6}a^3$ is less than $\frac{1}{120} \times 10^{-5}$ which is less than a millionth part of $a$.

45. The following results may be proved in a similar manner.

The difference between 1 and $\cos a$ is less than $\frac{1}{2}a^2$,

.....$\cos a$ and $(1 - \frac{1}{2}a^2)$..... $\frac{1}{24}a^4$.

EXAMPLE. *Find the limiting value of* $\dfrac{\sin^2 ma}{1 - \cos na}$ *when* $a$ *is infinitely diminished.*

For $\sin ma$, write $ma - R_1a^3$; for $\cos na$, write $1 - \frac{1}{2}n^2a^2 + R_2a^4$. We know that $R_1$ is less than $\frac{1}{6}m^3$ and that $R_2$ is less than $\frac{1}{24}n^4$. Then

$$\frac{\sin^2 ma}{1 - \cos na} = \frac{(ma - R_1a^3)^2}{\frac{1}{2}n^2a^2 - R_2a^4} = \frac{(m - R_1a^2)^2}{\frac{1}{2}n^2 - R_2a^2};$$

hence, when $a$ is infinitely diminished the required limit is $\dfrac{2m^2}{n^2}$.

## EXAMPLES. XIX.

(1) Prove that when $a$ is not large, the difference between $(a - \frac{1}{6}a^3 + \frac{1}{120}a^5)$ and $\sin a$ is less than $\frac{1}{5040}a^7$.

(2) Prove that when $a$ is not large, the difference between $(1 - \frac{1}{2}a^2 + \frac{1}{24}a^4)$ and $\cos a$ is less than $\frac{1}{720}a^6$.

(3) Prove that $\sin \frac{18^0}{\pi} = \cdot 099833$.

(4) Prove that the value of $\sin 1^0$ coincides with that of the circular measure of $1^0$ at least as far as five places of decimals.

(5) Solve the equation $\sin(\frac{1}{4}\pi + \theta) = \cdot 71$, neglecting $\theta^3$ and higher powers of $\theta$.

(6) Given that $\sin 1' = \cdot 0002909$, calculate approximately the value of $\pi$.

(7) Given $\frac{\sin\theta}{\theta} = \frac{1013}{1014}$, prove $\theta = 4^0\,24'$ nearly.

(8) Find the value of $\dfrac{\sin^2 n\theta - \sin^2 m\theta}{1 - \cos p\theta}$ when $\theta = 0$.

(9) Evaluate $\dfrac{\sin^2 \sqrt{mn}\theta - \sin m\theta \,.\, \sin n\theta}{(1 - \cos m\theta)(1 - \cos n\theta)}$ when $\theta = 0$.

(10) Find the limit of $\theta^{-4}\left\{\dfrac{\sin\theta}{\theta} + \dfrac{e^{\theta} - e^{-\theta}}{2\theta} - 2\right\}$ when $\theta$ is infinitely diminished.

(11) Prove that (eight times the chord of half a small circular arc minus the chord of the whole arc) divided by three, is equal to the length of the arc, nearly.

(12) Prove by induction

$$\tan n\theta = \frac{n\tan\theta - \dfrac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \ldots\ldots}{1 - \dfrac{n(n-1)}{1.2}\tan^2\theta + \ldots\ldots}.$$

(13) $\sin(\theta + h) = \sin\theta + h\cos\theta - \dfrac{h^2}{\lfloor 2}\sin\theta - \dfrac{h^3}{\lfloor 3}\cos\theta + \text{etc.}$

EXPANSION OF $(2\cos\alpha)^n$ AND OF $(2\sin\alpha)^n$.

46. The following notation will be found convenient.

**Def.** **cosh** $x$ stands for $\frac{e^x+e^{-x}}{2}$ and **sinh** $x$ for $\frac{e^x-e^{-x}}{2}$.

**cosh** $x$ and **sinh** $x$ are abbreviations respectively for the words **hyperbolic cosine of** $x$ and **hyperbolic sine of** $x$.

We shall use the notation $\cosh^2 x$ for $(\cosh x)^2$, etc.

## EXAMPLES. XX.

Prove the following statements : [Compare Examples XII.]

(1) $\cosh^2 x-\sinh^2 x=1$.

(2) $\cosh 2x=\cosh^2 x+\sinh^2 x=2\cosh^2 x-1=1+2\sinh^2 x$.

(3) $\cosh 3x=4\cosh^3 x-3\cosh x$. (4) $\cosh x=\cosh(-x)$.

(5) $\sinh 3x=3\sinh x+4\sinh^3 x$. (6) $\sinh x=-\sinh(-x)$.

(7) $\cosh(x+y)\,.\,\cosh(x-y)=\cosh^2 x+\sinh^2 y$
$=\cosh^2 y+\sinh^2 x$.

(8) $2\cosh nx\,.\,\cosh x=\cosh(n+1)x+\cosh(n-1)x$.

(9) $2\sinh nx\,.\,2^2\sinh^2 x=2\sinh(n+2)x-4\sinh nx$
$+2\sinh(n-2)x$.

(10) $2\cosh nx\,.\,2^2\sinh^2 x=2\cosh(n+2)x-4\cosh nx$
$+2\cosh(n-2)x$.

(11) $\cosh nx-\cos n\alpha=2\cosh(n-1)x\{\cosh x-\cos\alpha\}$
$+2\cos\alpha\{\cosh(n-1)x-\cos(n-1)\alpha\}-\{\cosh(n-2)x-\cos(n-2)\alpha\}$.

47. *To prove, when* n *is a positive integer, that* $2^n\cos^n\alpha$ *can be expressed in terms of* $\cos n\alpha$, $\cos(n-2)\alpha$, etc.; *that* $2^n\cosh^n x$ *can be expressed in terms of* $\cosh nx$, $\cosh(n-2)x$, etc.; *and that the two expressions are the* **same** *in form.*

We have, when $n$ is a positive integer,

$$2\cos n\alpha \,.\, 2\cos\alpha = 2\cos(n+1)\alpha + 2\cos(n-1)\alpha. \qquad \text{I.}$$

$$2\cosh nx \,.\, 2\cosh x = 2\cosh(n+1)x + 2\cosh(n-1)x. \qquad \text{II.}$$

In I. put $n=1$, and we obtain

$$2^2\cos^2\alpha = 2\cos 2\alpha + 2.$$

Multiply each side of the result by $2\cos\alpha$, then

$$2^3\cos^3\alpha = 2\cos 2\alpha \,.\, 2\cos\alpha + 4\cos\alpha.$$

But by I. $2\cos 2\alpha \,.\, 2\cos\alpha = 2\cos 3\alpha + 2\cos\alpha$,

hence $$2^3\cos^3\alpha = 2\cos 3\alpha + 6\cos\alpha.$$

Multiplying each side of this result by $2\cos\alpha$, and again making use of I., we have

$$2^4\cos^4\alpha = 2\cos 4\alpha + 8\cos 2\alpha + 6.$$

By multiplying each side of this result by $2\cos\alpha$, and making use of I. on the right-hand side, we can obtain an expression for $2^5\cos^5\alpha$ in the required form. And so on.

By continuing this process we could obtain an expression in the required form for $2^n\cos^n\alpha$ where $n$ is any positive integer.

Again, by making use of II. in the same manner we could obtain an expression in the required form for $2^n\cosh^n x$.

Also, since the *process is the same in each case*, the two resulting expressions are the *same* in form.

48. The expansion of $2^n\cosh^n x$ can be found as follows:

$$2^n\cosh^n x = (e^x + e^{-x})^n = e^{nx} + e^{-nx} + n\left(e^{(n-2)x} + e^{-(n-2)x}\right) + \text{etc.}$$

$$= 2\cosh nx + n \,.\, 2\cosh(n-2)x + \frac{n(n-1)}{1\,.\,2}\,2\cosh(n-4)x + \text{etc.}$$

Therefore also by Art. 47

$$2^n\cos^n\alpha = 2\cos n\alpha + n \,.\, 2\cos(n-2)\alpha + \frac{n(n-1)}{1\,.\,2}\,2\cos(n-4)\alpha + \text{etc.}$$

As in Art. 37, when $n$ is even, the last term does not contain $\cosh x$, and in this term the factor 2 is to be omitted.

*49. *To prove, when* n *is* **odd**, *that* $(-1)^{\frac{n-1}{2}} 2^n \sin^n a$ *can be expressed in terms of* $\sin na$, $\sin(n-2)a$, etc.; *that* $2^n \sinh nx$ *can be expressed in terms of* $\sinh nx$, $\sinh(n-2)x$, etc.; *and that the two expressions are the* **same** *in form.*

We have, when $n$ is a positive integer $> 2$,

$$2 \sin na\,(-1)\,2^2 \sin^2 a = 2 \sin(n+2)a - 4 \sin na + 2 \sin(n-2)a. \quad \text{I.}$$

$$2 \sinh nx \,.\, 2^2 \sinh^2 x = 2 \sinh(n+2)x - 4 \sinh nx + 2 \sinh(n-2)x. \quad \text{II.}$$

We have also

$$(-1)\,2^3 \sin^3 a = 2 \sin 3a - 4 \sin a.$$

We proceed as in Art. 47. Multiply each side of this result by $-2^2 \sin^2 a$, and we obtain by the aid of I.

$$(-1)^2 2^5 \sin^5 a = 2 \sin 5a - 10 \sin 3a + 20 \sin a.$$

Multiplying again by $-2^2 \sin^2 a$, we could obtain by the aid of I. an expression in the required form for $(-1)^3 2^7 \sin^7 a$.

By continuing this process we could obtain an expression in the required form for $(-1)^{\frac{n-1}{2}} 2^n \sin^n a$, where $n$ is any odd positive integer.

Again, since $2^3 \sinh^3 x = 2 \sinh 3x - 4 \sinh x$, by making use of II. in the same manner we could obtain an expression in the required form for $2^n \sinh^n x$.

And since *the process* on the right hand *is the same* in each case, the resulting expressions are the *same* in form.

*50. We have $2^n \sinh^n x = (e^x - e^{-x})^n$ [*n odd*]

$$= e^{nx} - e^{-nx} - n\,(e^{(n-2)x} - e^{-(n-2)x}) + \text{etc.}$$

$$= 2 \sinh nx - n\, 2 \sinh (n-2)\, x + \frac{n\,(n-1)}{1\,.\,2}\, 2 \sinh (n-4)\, x - \text{etc.}$$

Therefore it follows by Art. 49, that when $n$ is odd,

$$(-1)^{\frac{n-1}{2}}\, 2^n \sin^n \alpha = 2 \sin n\alpha - n\,.\, 2 \sin (n-2)\, \alpha + \frac{n\,(n-1)}{1\,.\,2}\, 2 \sin (n-4)\, \alpha - \text{etc.}$$

*51. *To prove that, when* n *is* **even**,

$$(-1)^{\frac{n}{2}}\, 2^n \sin^n \alpha = 2 \cos n\alpha - n\,.\, 2 \cos (n-2)\, \alpha + \frac{n\,(n-1)}{1\,.\,2}\, 2 \cos (n-4)\, \alpha - \text{etc.}$$

We have, when $n$ is a positive integer $> 2$,

$2 \cos n\alpha\,.\,(-1)\, 2^2 \sin^2 \alpha = 2 \cos (n+2)\, \alpha - 4 \cos n\alpha + 2 \cos (n-2)\, \alpha$

$2 \cosh nx\,.\, 2^2 \sinh^2 x$

$$= 2 \cosh (n+2)\, x - 4 \cosh nx + 2 \cosh (n-2)\, x.$$

Following the argument of the last article, we have since $(-1)\, 2^2 \sin^2 \alpha = 2 \cos 2\alpha - 2$, and $2^2 \sinh^2 x = 2 \cosh 2x - 2$. And since $2^n \sinh^n x = (e^x - e^{-x})^n$ [n *even*]

$$= 2 \cosh nx - n2 \cosh (n-2)\, x + \frac{n\,(n-1)}{1\,.\,2}\, 2 \cosh (n-4)\, x - \text{etc.}$$

Therefore when $n$ is even $(-1)^{\frac{n}{2}}\, 2^n \sin^n \alpha$

$$= 2 \cos n\alpha - n2 \cos (n-2)\, \alpha + \frac{n\,(n-1)}{1\,.\,2}\, 2 \cos (n-4)\, \alpha - \text{etc.}$$

In the last term the factor 2 must be omitted.

[Cf. Art. 37.]

## *EXAMPLES. XXI.

(1) Prove that $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

(2) $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$.

(3) $128\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$.

(4) Write down the last term in the expansion in multiples of $\cos\theta$, of

(i) $2^{2n}\cos^{2n}\theta$. (ii) $2^{2n+1}\cos^{2n+1}\theta$. (iii) $2^{4n+2}\sin^{4n+2}\theta$.

(5) Any general formula expressed in cosines is also true in hyperbolic cosines.

(6) Any general formula expressed in cosines or in squares of sines will be true in hyperbolic cosines and sines if we write $-\sinh^2\theta$ for $\sin^2\theta$.

52. *To prove, when* n *is a positive integer, that cos* n$\alpha$ *can be expressed in powers of cos* $\alpha$; *that cosh* nx *can be expressed in powers of cosh* x; *and that the two expressions are the* **same** *in form.*

We have, when $n$ is a positive integer

$$2\cos(n+1)\alpha = 4\cos n\alpha \,.\, \cos\alpha - 2\cos(n-1)\alpha. \qquad \text{(I.)}$$

$$2\cosh(n+1)x = 4\cosh nx \,.\, \cosh x - 2\cosh(n-1)x. \qquad \text{(II.)}$$

In I. put $n=1$, and we obtain

$$2\cos 2\alpha = 4\cos^2\alpha - 2.$$

Next put $n=2$, and using this last result, we have

$$2\cos 3\alpha = 8\cos^3\alpha - 6\cos\alpha.$$

Put $n=3$, then using the last two results we have

$$2\cos 4\alpha = 16\cos^4\alpha - 16\cos^2\alpha + 2.$$

Next put $n=4$, then by the aid of the last two results, we can obtain an expression for $2\cos 5\alpha$ in powers of $\cos\alpha$; and so on.

By proceeding in this way we could obtain an expression in the required form for $2\cos n\alpha$ when $n$ is any positive integer.

Again, by making use of (II.) in the same manner we could obtain an expression in the required form for $2\cosh nx$.

Also, since the *process is the same in each case*, the two resulting expressions are the *same* in form.

EXAMPLE. *Prove that* $\cosh \mathrm{n}x - \cos \mathrm{n}\alpha$ *is divisible by* $\cosh \mathrm{x} - \cos \alpha$ *when* n *is a positive integer.*

From the above we have

$$\cosh nx = A_n \cosh^n x + A_{n-2} \cosh^{n-2} x + \text{etc.}$$

$$\cos n\alpha = A_n \cos^n \alpha + A_{n-2} \cos^{n-2} \alpha + \text{etc.}$$

the *coefficients* in the two expressions being the *same*.

Hence by subtraction

$$\cosh nx - \cos n\alpha = A_n (\cosh^n x - \cos^n \alpha) + A_{n-2} (\cosh^{n-2} x - \cos^{n-2} \alpha) + \ldots$$

and each term in this expression is divisible by $\cosh x - \cos \alpha$; [for $y^n - z^n$ is divisible by $y - z$ when $n$ is a positive integer] therefore also $\cosh nx - \cos n\alpha$ is divisible by $\cosh x - \cos \alpha$.

## EXAMPLES. XXII.

(1) Prove that $2\cos 6\theta = 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2$; also that $e^{6x} + e^{-6x} = (e^x + e^{-x})^6 - 6(e^x + e^{-x})^4 + 9(e^x + e^{-x})^2 - 2$.

(2) Divide $\cosh 6x - \cos 6\theta$ by $\cosh x - \cos \theta$.

(3) Prove that $x^n + x^{-n} - 2\cos n\alpha$

is divisible by $x + x^{-1} - 2\cos \alpha$ when $n$ is a positive integer.

*53. We can find the law of the coefficients in the expansion for $2\cos nx$, as follows: since

$$(1-e^x z)(1-e^{-x}z) = 1 - z(e^x + e^{-x}) + z^2 = 1 - z\,2\cos x + z^2,$$

$$\therefore \quad \log(1-e^x z) + \log(1-e^{-x}z) = \log\{1 - z(2\cos x - z)\};$$

$$\therefore \quad e^x z + \tfrac{1}{2}e^{2x}z^2 + \tfrac{1}{3}e^{3x}z^3 + \text{etc.} + e^{-x}z + \tfrac{1}{2}e^{-2x}z^2 + \tfrac{1}{3}e^{-3x}z^3 + \text{etc.}$$

$$= z(2\cos x - z) + \tfrac{1}{2}z^2(2\cos x - z)^2 + \tfrac{1}{3}z^3(2\cos x - a)^3 + \text{etc.}$$

In this *identity* the coefficients of $z^n$ on each side must be equal.

On the left-hand side the coefficient of $z^n$ is $\frac{1}{n}(e^{nx} + e^{-nx})$, that is $\frac{2}{n}\cos nx$. The coefficient of $z^n$ in $\frac{1}{n}z^n(2\cos x - z)^n$ is $\frac{1}{n}2^n\cos^n x$. The coefficient of $z^n$ in $\frac{1}{n-1}z^{n-1}(2\cos x - z)^{n-1}$ is $-\frac{1}{n-1}(n-1)\,2^{n-2}\cos^{n-2}x$, and so on. Thus we get

$$\frac{2}{n}\cos nx = \frac{2^n}{n}\cos^n x - \frac{1}{n-1}(n-1)\,2^{n-2}\cos^{n-2}x + \frac{1}{n-2}\cdot\frac{(n-2)(n-3)}{1\,.\,2}\,2^{n-4}\cos^{n-4} - \frac{1}{n-3}\,\frac{(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3}\,2^{n-6}\cos^{n-6}x + \text{etc.}$$

Therefore also $2\cos n\alpha = 2^n\cos^n\alpha - n\,.\,2^{n-2}\cos^{n-2}\alpha +$

$$n\,.\,\frac{(n-3)}{1\,.\,2}\,2^{n-4}\cos^{n-4}\alpha - n\,.\,\frac{(n-4)(n-5)}{1\,.\,2\,.\,3}\,2^{n-6}\cos^{n-6}\alpha + \text{etc.}$$

*54. This result may be transformed into a more symmetrical form as follows. The general term is

$$(-1)^r\,\frac{n(n-r-1)\ldots(n-2r+1)}{\lfloor r}\,(2\cos\theta)^{n-2r}.$$

I. Let $n$ be **even**; let $2m$ stand for $n$.

Then $n-2r$ is even; let $2p$ stand for $n-2r$.

The general term may be written

$$(-1)^{m-p}\frac{n(m+p-1)(m+p-2)\ldots(2p+1)}{\lfloor m-p}2^{2p}\cos^{2p}\theta$$

[where $p$ is to have all integral values from 0 to $m$]

$$=(-1)^{m-p}\frac{n\lfloor m+p-1}{\lfloor m-p\,\lfloor 2p}2^{2p}\cos^{2p}\theta$$

$$=(-1)^{m-p}\frac{n(m+p-1)(m+p-2)\ldots(m-p+1)}{\lfloor 2p}2^{2p}\cos^{2p}\theta$$

$$=(-1)^{m-p}\times$$

$$\frac{n(2m+2p-2)(2m+2p-4)\ldots(2m)\ldots(2m-2p+2)}{\lfloor 2p}2\cos^{2p}\theta$$

[for there are $2p-1$ terms in the series $(m+p-1)$, $(m+p-2)$ $\ldots(m-p+1)$]

$$=(-1)^{m-p}\frac{\{n^2-(2p-2)^2\}\{n^2-(2p-4)^2\}\ldots(n^2)}{\lfloor 2p}2\cos^{2p}\theta.$$

Hence putting for $p$ the values 0, 1, 2 ... we have, $2\cos n\theta$

$$=(-1)^{\frac{n}{2}}\,2\left\{1-\frac{n^2}{1.2}\cos^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta-\text{etc.}\right\}.$$

II. Let $n$ be **odd**, then by putting $2m+1$ for $n$, and making a similar transformation, we shall obtain

$$2\cos n\theta=(-1)^{\frac{n-1}{2}}\times$$

$$2\left\{n\cos\theta-\frac{n(n^2-1^2)}{1.2.3}\cos^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\cos^5\theta-\text{etc.}\right\}.$$

*55. The following is an illustration of an important method.

Suppose that we have a general theorem such as

$$\cos n\theta = A_0 + A_1 \cos\theta + A_2 \cos^2\theta + A_3 \cos^3\theta + \text{etc.}\ldots\ldots\text{(I)},$$

which is true for all values of $\theta$. For $\theta$ put $\theta + h$ and we have

$$\cos n\theta \,.\, \cos nh - \sin n\theta \,.\, \sin nh$$
$$= A_0 + A_1 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)$$
$$+ A_2 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)^2$$
$$+ A_3 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)^3 + \text{etc.}$$

For $\cos h$ we may write $1 - Rh^2$ and for $\sin h$ we may write $h - R'h^3$, where $R$ and $R'$ are both finite when $h = 0$; hence we obtain [Art. 44.]

$$\cos n\theta - n^2h^2R\cos n\theta - nh\sin n\theta + n^2h^3R'\sin n\theta$$
$$= A_0 + A_1\cos\theta + A_2\cos^2\theta + \text{etc.}$$
$$- h\{A_1\sin\theta + 2A_2\sin\theta\,.\,\cos\theta$$
$$+ 3A_3\sin\theta\,.\,\cos^2\theta + 4A_4\sin\theta\,.\,\cos^3\theta + \ldots\}$$
$$+ \text{terms containing higher powers of } h \ldots\ldots\ldots\text{(II)}.$$

This result is true for all values of $h$, and remembering I. we see that it is divisible by $h$. Dividing by $h$ we get a result which is true for all values of $h$, and is therefore true in the limit when $h = 0$. Proceeding to the limit we obtain

$$+ n\sin n\theta = A_1\sin\theta + 2A_2\sin\theta\cos\theta + 3A_3\sin\theta\cos^2\theta + \text{etc.}$$

*56. The student who is familiar with the methods of the Differential Calculus will observe that the above result may be obtained by differentiating each side of the equation (I).

*57. Applying this result to the series of Art. 54 we have, when $n$ is **even**

$$2\sin n\theta = (-1)^{\frac{n}{2}+1} 2\sin\theta \times \left\{ n\cos\theta - \frac{n(n^2-2^2)}{1.2.3}\cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4.5}\cos^5\theta - \text{etc.} \right\},$$

when $n$ is **odd**

$$2\sin n\theta = (-1)^{\frac{n-1}{2}} 2\sin\theta \times \left\{ 1 - \frac{(n^2-1^2)}{1.2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4}\cos^4\theta - \text{etc.} \right\}.$$

*58. Hence we have the series:

I. ($n$ **even**), $(-1)^{\frac{n}{2}}\cos n\theta = 1 - \frac{n^2}{1.2}\cos^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta -$

II. ,, $(-1)^{\frac{n}{2}+1}\sin n\theta = \sin\theta\left\{ n\cos\theta - \frac{n(n^2-2^2)}{1.2.3}\cos^3\theta + \right\}$

III. ($n$ **odd**), $(-1)^{\frac{n-1}{2}}\cos n\theta = n\cos\theta - \frac{n(n^2-1^2)}{1.2.3}\cos^3\theta + \ldots$

IV. ,, $(-1)^{\frac{n-1}{2}}\sin n\theta = \sin\theta\left\{ 1 - \frac{n^2-1^2}{1.2}\cos^2\theta + \ldots \right\}.$

*59. In each of the above formulæ put $\frac{\pi}{2} - \theta$ for $\theta$, then

I. ($n$ **even**), $\cos n\theta = 1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \ldots$

II. ,, $\sin n\theta = \cos\theta\left\{ n\sin\theta - \frac{n(n^2-2^2)}{1.2.3}\sin^3\theta + \ldots \right\}$

III. ($n$ **odd**), $\sin n\theta = n\sin\theta - \frac{n(n^2-1^2)}{1.2.3}\sin^3\theta + \ldots$

IV. ,, $\cos n\theta = \cos\theta\left\{ 1 - \frac{n^2-1^2}{1.2}\sin^2\theta + \ldots \right\}.$

*60. In the following example an independent proof is given of the result of Art. 54.

EXAMPLE. *To expand* $\cos n\theta$ *in ascending powers of* $\sin\theta$.

From Art. 19, we have when $n$ is *even*

$$\cos n\theta = 1 + A_2 \sin^2\theta + A_4 \sin^4\theta + \text{etc.} \quad \text{.....I.}$$

The constant term is 1, because when $\theta=0$, $\cos n\theta=1$, and $\sin\theta=0$.

For $\theta$ write $\theta+h$ and we have

$$\cos n\theta \cos nh + \sin n\theta \sin nh$$
$$= 1 + A_2(\cos\theta\cos h + \sin\theta\sin h)^2 + A_4(\cos\theta\cos h + \sin\theta\sin h)^4 + \ldots$$

For $\cos nh$, write $1 - \frac{1}{2}n^2h^2 + \text{etc.}$, for $\sin nh$, write $nh - \frac{1}{6}n^3h^3 + \text{etc.}$, and substitute similar expressions for $\cos h$ and $\sin h$.

In the result we may equate the coefficients of $h^2$.

On the left-hand side the coefficient of $h^2$ is $-\frac{1}{2}n^2\cos^2 n\theta$.

In the term $A_{2r}(\sin\theta + h\cos\theta - \frac{1}{2}h^2\sin\theta - \text{etc.})^{2r}$ the coefficient of $h^2$ is

$$A_{2r}\left\{\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta\cos^2\theta - r\sin^{2r}\theta\right\}.$$

Hence

$$-\tfrac{1}{2}n^2\cos n\theta = A_2\{\cos^2\theta - \sin^2\theta\} + A_4\{2.3\sin^2\theta\cos^2\theta - 2\sin^4\theta\} + \ldots$$
$$+ A_{2r}\left\{\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta\cos^2\theta - r\sin^{2r}\theta\right\} + \ldots$$

In this writing $1-\sin^2\theta$ for $\cos^2\theta$ we obtain as the coefficient of $\sin^{2r}\theta$

$$-A_{2r}\left\{\frac{2r(2r-1)}{1.2} + r\right\} + A_{2r+2}\left\{\frac{(2r+2)(2r+1)}{1.2}\right\}.$$

This then gives another form for the expansion of $\cos n\theta$ in ascending powers of $\sin^2\theta$.

The coefficients of these two series are the same. Hence

$$-\tfrac{1}{2}n^2A_{2r} = -A_{2r}\left\{\frac{2r(2r-1)}{1.2} + r\right\} + A_{2r+2}\frac{(2r+2)(2r+1)}{1.2},$$

or

$$A_{2r+2} = -\frac{n^2-(2r)^2}{(2r+1)(2r+2)}A_{2r}.$$

Putting for $r$ the values 1, 2, 3 ... in succession we obtain

$$A_2 = -\frac{n^2}{1.2} A_0 = -\frac{n^2}{1.2},$$

$$A_4 = -\frac{n^2-2^2}{3.4} A_2 = \frac{n^2(n^2-2^2)}{1.2.3.4},$$

and so on.

Thus $\cos n\theta = 1 - \frac{n^2}{1.2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \text{etc.}$

[The same result would be obtained by comparing the series I. with that obtained by equating the second differentials of each side.]

## *EXAMPLES. XXIII.

Prove the following statements :

(1) $\cos 4\theta = 1 - 8\cos^2\theta + 8\cos^4\theta.$

(2) $-\cos 6\theta = 1 - 18\cos^2\theta + 48\cos^4\theta - 32\cos^6\theta.$

(3) $\cos 9\theta = 9\cos\theta - 120\cos^3\theta + 432\cos^5\theta - 576\cos^7\theta + 2^9\cos^9\theta.$

(4) $\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta.$

(5) $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \text{etc.}$

(6) $\sin(x+h) = \sin x + h\cos x - \frac{h^2}{\lfloor 2}\sin x - \frac{h^3}{\lfloor 3}\cos x + \text{etc.}$

(7) $\cosh(x+h) = \cosh x + h\sinh x + \frac{h^2}{\lfloor 2}\cosh x + \text{etc.}$

(8) $\sinh(x+h) = \sinh x + h\cosh x + \frac{h^2}{\lfloor 2}\sinh x + \text{etc.}$

(9) $\log(x+h) = \log x + \frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3} \ldots\ldots$

(10) If $\sin na = A_1\sin a = A_3\sin^3 a + \ldots$, then

$$n\cos na = A_0\cos a + 3A_3\cos a\sin^2 a + \text{etc.}$$

(11) Prove that if $\cos^n a = A_n\cos na + A_{n-2}\cos(n-2)a + \text{etc.}$ then $n\sin a \,.\, \cos^{n-1} a = nA_n\sin na + (n-2)A_{n-2}\sin(n-2)a + \text{etc.}$

(12) Prove that $2^n \sin a \cos^{n-1} a = 2 \sin na + 2(n-2) \sin (n-2)a + (n-1)(n-4) \cos (n-4)a + \text{etc.}$

(13) Prove that

$$2 \sin na = 2^n \sin a \,.\, \cos^{n-1} a - 2^{n-2}(n-2) \sin a \,.\, \cos (n-3)a + (n-3)(n-4)\, 2^{n-5} \sin a \,.\, \cos (n-5)a - \text{etc.}$$

(14) Assuming when $n$ is odd that

$$\sin n\theta = n \sin \theta + A_3 \sin^2 \theta + A_5 \sin^5 \theta + \ldots$$

prove as in the Example on page 54, that

$$\sin n\theta = n \sin \theta - \frac{n(n^2-1)}{1.2.3} \sin^2 \theta + \frac{n(n^2-1)(n^2-3^2)}{1.2.3.4.5} \sin^5 \theta \ldots$$

(15) If $\sinh nx = A_1 \sinh x + A_3 \sinh^3 x + \text{etc.}$, then

$$n \cosh nx = A_1 \cosh x + 3A_3 \cosh x \sinh^2 x + \ldots$$

*61. Consider the equation ($n$ **even**)

$$2^n \cos^n \theta - n\, 2^{n-2} \cos^{n-2} \theta + \frac{n(n-3)}{1.2} 2^{n-4} \cos^{n-4} \theta + \text{etc.}$$

$$+ (-1)^{\frac{n}{2}} \frac{n^2(n^2-2^2)}{\underline{4}} 2 \cos^4 \theta - (-1)^{\frac{n}{2}} \frac{n^2}{1.2} 2 \cos^2 \theta + (-1)^{\frac{n}{2}} 2 = 2 \cos na.$$

This is an equation of the $n$th degree, one of whose roots is $a$; and since $\cos na = \cos n\left(a + \frac{2r\pi}{n}\right)$, the other roots are found by giving $r$ the values 1, 2, 3... $(n-1)$ in the expression $\cos\left(a + \frac{2r\pi}{n}\right)$. This result is useful in solving problems on symmetrical functions of the series

$$\cos a,\ \cos\left(a + \frac{2\pi}{n}\right),\quad \cos\left(a + \frac{4\pi}{n}\right),\ \text{etc.}$$

EXAMPLE. Find the sum of the series

$$\sec^2\alpha+\sec^2\left(\alpha+\frac{2\pi}{n}\right)+\sec^2\left(\alpha+\frac{4\pi}{n}\right)+\ \ldots \text{ to } n \text{ terms.}$$

That is, find the sum of the squares of the reciprocals of the roots of the above equation.

Hence, if $S_0$, $S_1$, $S_2$, etc. are the coefficients of $\cos^n\theta$, $\cos^{n-1}\theta$, etc. in the above equation, the required sum $=\left(\frac{S_{n-1}}{S_n}\right)^2-\frac{2S_{n-2}}{S_n}$;

and when $n$ is even

$$-S_n=\{2\cos n\alpha-(-1)^{\frac{n}{2}}2\},\qquad S_{n-1}=0,\qquad S_{n-2}=-(-1)^{\frac{n}{2}}n^2.$$

The required sum is $\dfrac{n^2}{1-(-1)^{\frac{n}{2}}\cos n\alpha}$. When $n$ is odd, the sum is $\dfrac{n^2}{\cos^2 n\alpha}$.

## *EXAMPLES. XXIV.

Find the value of the following expressions, in which $n\phi=2\pi$.

(1) $\cos\alpha\cos(\alpha+\phi)\cos(\alpha+2\phi)\ldots\cos\{\alpha+(n-1)\phi\}$.

(2) $\sec\alpha+\sec(\alpha+\phi)+\sec(\alpha+2\phi)+\ldots$ to $n$ terms.

(3) $\sin\alpha\sin(\alpha+\phi)\sin(\alpha+2\phi)+\ldots$ to $n$ factors.

(4) $\operatorname{cosec}^2\alpha+\operatorname{cosec}^2(\alpha+\phi)+\operatorname{cosec}^2(\alpha+2\phi)\ldots$ to $n$ terms.

(5) $\tan^2\alpha+\tan^2(\alpha+\frac{1}{2}\phi)+\tan^2(\alpha+\phi)+$ to $n$ terms.

(6) $\tan\alpha+\tan(\alpha+\frac{1}{2}\phi)+\tan(\alpha+\phi)+$ to $n$ terms.

(7) $\cot\alpha+\cot(\alpha+\frac{1}{2}\phi)+\cot(\alpha+\phi)+\ldots$ to $n$ terms.

*62. *To find the Quadratic Factors of* $x^n - 2\cos n\alpha + x^{-n}$, when $n$ is a whole number. [Compare Arts. 38, 52.]

The following is an identity:

$$x^n - 2\cos n\alpha + x^{-n} = \{x^{n-1} + x^{-(n-1)}\}\{x - 2\cos\alpha + x^{-1}\}$$
$$+ 2\cos\alpha\{x^{n-1} - 2\cos(n-1)\alpha + x^{-(n-1)}\}$$
$$- \{x^{n-2} - 2\cos(n-2)\alpha + x^{-(n-2)}\}.$$

Let $f(n)$ stand for $x^n - 2\cos n\alpha + x^{-n}$.

Then the above identity may be written

$$f(n) = \{x^{n-1} + x^{-(n-1)}\}f(1) + 2\cos\alpha f(n-1) - f(n-2).$$

Now from this it is clear that if $f(1)$ divides both $f(n-1)$ and $f(n-2)$, it must also divide $f(n)$.

But $f(1)$ does divide $f(1)$ and $f(2)$. Therefore $f(1)$ divides $f(3)$; and therefore $f(n)$, when $n$ is any positive integer. That is, $(x - 2\cos\alpha + x^{-1})$ is a factor of $(x^n - 2\cos n\alpha + x^{-n})$.

Again, it follows that $x - 2\cos\left(\alpha + \frac{2r\pi}{n}\right) + x^{-1}$ is a factor of $x^n - 2\cos(n\alpha + 2r\pi) + x^{-n}$, that is of $x^n - 2\cos n\alpha + x^{-n}$. Hence we get that the $n$ factors of $x^n - 2\cos n\alpha + x^{-n}$ are

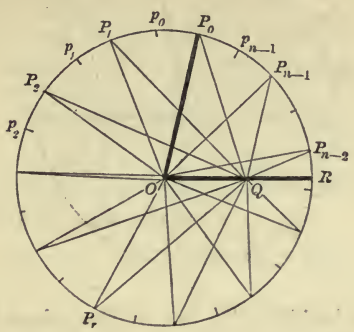
$$\{x - 2\cos\alpha + x^{-1}\}\left\{x - 2\cos\left(\alpha + \frac{2\pi}{n}\right) + x^{-1}\right\}\ldots\ldots$$

$n$ factors up to $\left\{x - 2\cos\left(\alpha + \frac{2(n-1)\pi}{n}\right) + x^{-1}\right\}$.

63. Writing $\frac{x}{a}$ for $x$ in the result of Arts. 38, 52, or 62, and simplifying, we get

$$x^{2n} - a^n x^n \,.\, 2\cos n\alpha + a^{2n} = \{x^2 - ax \,.\, 2\cos\alpha + a^2\}$$
$$\times \left\{x^2 - ax \,.\, 2\cos\left(\alpha + \frac{2\pi}{n}\right) + a^2\right\} \times \text{etc. } n \text{ factors.}$$

64. This result may be interpreted geometrically.




Let $OR$ be the initial line; with centre $O$ and radius equal to $x$ describe a circle; let $ROP_0$ be the angle $\alpha$.

Divide the whole circumference, starting from $P_0$, into $n$ equal parts, $P_0P_1$, $P_1P_2$, ... $P_{n-1}P_0$.

Let $OQ$ be equal to $a$, so that $Q$ is any point in $OR$ or in $OR$ produced, then

$$QP_0^2 = OP_0^2 + OQ^2 - OP_0 \,.\, OQ \,.\, 2\cos ROP_0 = x^2 + a^2 - ax \,.\, 2\cos\alpha$$

$$QP_1^2 = OP_1^2 + OQ^2 - OP_1 OQ \,.\, 2\cos ROP_1$$

$$= x^2 + a^2 - ax \,.\, 2\cos\left(\alpha + \frac{2\pi}{n}\right) \text{ and so on.}$$

Hence the result of Art. 56 may be written

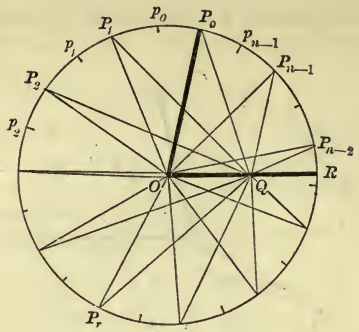
$$OP_0^{2n} - OP_0^n \,.\, OQ^n 2\cos nROP_0 + OQ^{2n} = QP_0^2 \,.\, QP_1^2 \,.\, QP_2^2 \ldots QP_{n-1}^2.$$

This result is **De Moivre's property of the circle.**

65. Some particular cases of the above should be noticed.

When $Q$ coincides with $R$, $a = x$, and the above becomes

$$OR^n \,.\, 2\sin\left(\tfrac{1}{2}n \,.\, ROP_0\right) = RP_0 \,.\, RP_1 \,.\, RP_2 \ldots RP_{n-1} \ldots \text{(I)}.$$




Again if $R$ coincides with one of the points $P$, then $\alpha$ is a multiple of $\frac{2\pi}{n}$, and $n\alpha$ is a multiple of $2\pi$ and we have

$$(OR^n - OQ^n)^2 = QP_0^2 \cdot QP_1^2 \cdot QP_2^2 \ldots QP_{n-1}^2;$$

$$\therefore\ OR^n - OQ^n = QP_0 \cdot QP_1 \cdot QP_2 \ldots QP_{n-1} \ldots\ldots\text{(II)}.$$

Now if the arcs $P_0P_1$, $P_1P_2$,... are bisected in points $p_0$, $p_1$, $p_2 \ldots p_{n-1}$ respectively, we have by what has just been proved

$$OR^{2n} - OQ^{2n} = QP_0 \cdot Qp_0 \cdot QP_1 \cdot Qp_2 \ldots QP_{n-1} \cdot Qp_{n-1}.$$

Therefore, by division

$$OR^n + OQ^n = Qp_0 \cdot Qp_1 \cdot Qp_2 \ldots Qp_{n-1} \ldots\ldots\text{(III)}.$$

The student should notice carefully that in (I) $Q$ lies somewhere *on* the circumference; in (II) $OQ$ or $OQ$ produced, passes through *one of the points* $P_0P_1 \ldots$; in (III) $OQ$ or $OQ$ produced, passes through *the middle point of one of the arcs* $p_0p_1$, $p_1p_2$, etc.

(II) and (III) are **Cote's properties of the circle.**

## *MISCELLANEOUS EXAMPLES. XXV.

(1) Prove Euler's Formula, viz.

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2^2} \cdot \cos\frac{\theta}{2^3} \ldots \text{ ad inf.}$$

and deduce from it that $\sin\theta$ is less than $\theta - \frac{1}{6}\theta^3$.

(2) $AB$ is the diameter of a circle and $Q_0$ any point on the circumference; $Q_1$, $Q_2$, $Q_3$,... are the points of bisection of the arcs $AQ_0$, $AQ_1$, $AQ_2$,... prove that

$$BQ_1 \cdot BQ_2 \cdot BQ_3 \ldots BQ_n = OA^n \cdot \frac{AQ_0}{AQ_n}.$$

(3) Find the limit of $(\cos\theta)^{\cot\theta}$ when $\theta = 0$.

(4) Find the limit of $\log_{\tan\theta} \sin\theta$ when $\theta = 0$.

(5) Of what order is the error when $\dfrac{3\sin\theta}{2+\cos\theta}$ is substituted for $\theta$?

(6) Prove that $2\cos n\phi - 2\cos n\theta = 2^n(\cos\phi - \cos\theta) \times$

$$\left\{\cos\phi - \cos\left(\frac{2\pi}{n} + \theta\right)\right\} \ldots\ldots \left\{\cos\phi - \cos\left(\frac{2n-1}{n}\pi + \theta\right)\right\}.$$

(7) $\tan^{-1}\left(\dfrac{\tan 2\theta + \tanh 2x}{\tan 2\theta - \tanh 2x}\right) + \tan^{-1}\left(\dfrac{\tan\theta - \tanh x}{\tan\theta + \tanh x}\right)$

$$= \tan^{-1}(\cot\theta \coth x)$$

where $\tanh x = \dfrac{\sinh x}{\cosh x}$ and $\coth x = \dfrac{\cosh x}{\sinh x}$.

(8) Expand $\cos^{4n}\theta + \sin^{4n}\theta$ in a series of cosines of multiples of $\theta$.

(9) Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{27}.$$

(10) Form the equation whose roots are

$$\tan^2\frac{\pi}{11},\ \tan^2\frac{2\pi}{11},\ \tan^2\frac{3\pi}{11},\ \tan^2\frac{4\pi}{11},\ \tan^2\frac{5\pi}{11}.$$

## CHAPTER V.

### On the Summation of Trigonometrical Series.

66. There are two methods peculiarly applicable to the Summation of Trigonometrical Series.

### FIRST METHOD.

67. Sometimes each term of a series may be transformed into the **difference** of two quantities.

Example 1. *To sum the series*

$$\sin\alpha+\sin(\alpha+\delta)+\sin(\alpha+2\delta)+\ \dots\ +\sin\{\alpha+(n-1)\delta\}.$$

We have $2\sin\alpha\,.\,\sin\frac{1}{2}\delta=\cos(\alpha-\frac{1}{2}\delta)-\cos(\alpha+\frac{1}{2}\delta),$

$2\sin(\alpha+\delta)\,.\,\sin\frac{1}{2}\delta=\cos(\alpha+\frac{1}{2}\delta)-\cos(\alpha+\frac{3}{2}\delta),$

$2\sin(\alpha+2\delta)\,.\,\sin\frac{1}{2}\delta=\cos(\alpha+\frac{3}{2}\delta)-\cos(\alpha+\frac{5}{2}\delta),$

.....

$$2\sin\{\alpha+(n-1)\delta\}\,.\,\sin\tfrac{1}{2}\delta=\cos\left(\alpha+\frac{2n-3}{2}\delta\right)-\cos\left(\alpha+\frac{2n-1}{2}\delta\right).$$

Therefore, if $S_n$ stands for the sum of $n$ terms, we obtain by addition

$$2S_n\,.\,\sin\tfrac{1}{2}\delta=\cos(\alpha-\tfrac{1}{2}\delta)-\cos\left(\alpha+\frac{2n-1}{2}\delta\right)$$

$$=2\sin\{\alpha+\tfrac{1}{2}(n-1)\delta\}\,.\,\sin\tfrac{1}{2}n\delta.$$

Therefore
$$S_n=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\delta\}\,.\,\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

Example 2. *To sum the series*

$$\cos\alpha+\cos(\alpha+\delta)+\cos(\alpha+2\delta)+\ \dots\ \cos\{\alpha+(n-1)\delta\}.$$

We have $2\cos\alpha\,.\,\sin\frac{1}{2}\delta=\sin(\alpha+\frac{1}{2}\delta)-\sin(\alpha\frac{1}{2}-\delta).$

Hence, proceeding as in Example 1, we obtain

$$S_n=\frac{\cos\{\alpha+\frac{1}{2}(n-1)\delta\}\,.\,\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

The results of these two examples are often useful. The student is advised to become familiar with them in *words*.

*The sum of* n *terms of a series of sines* (or cosines) *of angles in* A. P. *is equal to the sine* (or cosine) *of half the sum of the first and last angle, multiplied by the sine of* n *times half the difference, divided by the sine of half the difference.*

EXAMPLE 3. *To prove that if* $n\phi = 2\pi$, *then*

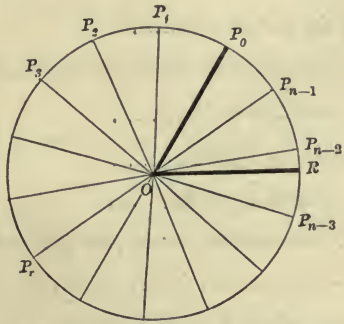
$$\sin\alpha + \sin(\alpha+\phi) + \sin(\alpha+2\phi) \ \ldots \ + \sin\{\alpha+(n-1)\phi\} = 0$$

*for all values of* $\alpha$.

In the result of Example 1, $\sin\frac{1}{2}n\delta$ occurs in the numerator, and $\sin\frac{1}{2}n\delta = \sin\frac{1}{2}n\phi = \sin\pi = 0$, and the denominator $\sin\frac{1}{2}\phi$ is *not* $= 0$. Therefore the sum of the series $= 0$. Similarly

$$\cos\alpha + \cos(\alpha+\phi) + \cos(\alpha+2\phi) + \ \ldots \ + \cos\{\alpha+(n-1)\phi\} = 0.$$

68. The results of Example 3 may be stated geometrically: Let $OR$ be the initial line and $ROP_0$ any angle, then if the whole circumference of a circle centre $O$ and radius $OR$, be divided into *n equal parts* $P_0P_1$, $P_1P_2$, etc. Then the sum of the sines (or of the cosines) of all the angles $ROP_0$, $ROP_1$ ... $ROP_{n-1}$ is zero†.




† This is an expression of the fact that the centre of gravity of equal particles placed at the points $P_0P_1$... is at the centre of the circle.

EXAMPLE 4. To sum the series

$$\sin^m \alpha + \sin^m (\alpha+\delta) + \sin^m (\alpha+2\delta) + \ \dots\ + \sin^m \{\alpha + (n-1)\,\delta\}.$$

This may be done by the aid of Arts. 37, 47.

Thus, if $m$ be even

$$2^m \sin^m \alpha = (-1)^{\frac{m}{2}} \{\cos m\alpha - n \cos (m-2)\,\alpha + \text{etc.}\}$$

and the required sum may be obtained from the known sum of the series $\{\cos m\alpha + \cos m\,(\alpha+\delta) + \cos m\,(\alpha+2\delta) + \text{etc.}\}$

$$+ \{\cos (m-2)\,\alpha + \cos (m-2)\,(\alpha+\delta) + \cos (m-2)\,(\alpha+2\delta) + \text{etc.}\}.$$

Similarly we may find the sum of the series

$$\cos^m \alpha + \cos^m (\alpha+\delta) + \cos^m (\alpha+2\delta) + \text{etc. to } n \text{ terms.}$$

## EXAMPLES. XXVI.

Sum the following series to $n$ terms.

(1) $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \dots\dots$

(2) $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \dots\dots$

(3) $\sin \alpha + \sin 4\alpha + \sin 7\alpha + \dots\dots$

(4) $\sin \alpha \,.\, \cos \alpha + \sin 2\alpha \,.\, \cos 2\alpha + \sin 3\alpha \,.\, \cos 3\alpha + \dots\dots$

(5) $\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots\dots$

(6) $\sin^3 \alpha + \sin^3 2\alpha + \sin^3 3\alpha + \dots\dots$

(7) $\cos^4 \alpha + \cos^4 2\alpha + \cos^4 3\alpha + \dots\dots$

(8) $\sin 2\alpha \,.\, \cos \alpha + \sin 3\alpha \,.\, \cos 2\alpha + \sin 4\alpha \,.\, \cos 3\alpha + \dots\dots$

(9) $\sin \alpha \,.\, \sin 2\alpha + \sin 2\alpha \,.\, \sin 3\alpha + \sin 3\alpha \,.\, \sin 4\alpha + \dots\dots$

(10) $\cos^3 \alpha + \cos^3 (\alpha+\delta) + \cos^3 (\alpha+2\delta) + \dots\dots$

(11) $\sin^4 \alpha + \sin^4 (\alpha+\delta) + \sin^4 (\alpha+2\delta) + \dots\dots$

(12) Solve the equation

$\sin\theta+\sin 2\theta+\sin 3\theta+$ etc. to $n$ terms $=\cos\theta+\cos 2\theta+\cos 3\theta+$ etc. to $n$ terms.

(13) Write down the value of series (10) and (11) when $n\delta=2\pi$.

(14) Prove that

$$\frac{\sin a+\sin 3a+\sin 5a+\ \ldots\ \text{to } n \text{ terms}}{\cos a+\cos 3a+\cos 5a+\ \ldots\ \text{to } n \text{ terms}}=\tan na.$$

(15) Prove that

$$\frac{\sin a+\sin(a+\delta)+\sin(a+2\delta)+\ \ldots\ \text{to } (2n-1) \text{ terms}}{\sin a+\sin(a+2\delta)+\sin(a+4\delta) \text{ to } n \text{ terms}}$$

is independent of $a$.

(16) Deduce from Ex. (1) the sum of the series

$$1+2+3+\ \ldots\ldots\ +n.$$

(17) Deduce from Ex. (6) the sum of the series

$$1^3+2^3+3^3+\text{ etc.}+n^3.$$

(18) Deduce from Ex. (9) the sum of the series

$$1.2+2.3+3.4+\text{etc.}+n(n+1).$$

(19) Sum the series $\sin a-\sin(a+\delta)+\sin(a+2\delta)-$ etc. to $n$ terms.

(20) Sum the series $\cos a-\cos(a+\delta)+\cos(a+2\delta)-$ etc. to $n$ terms.

(21) Prove that the series $\sin^m a+\sin^m(a+\phi)+$ etc. to $n$ terms, where $n\phi=2\pi$, is independent of $a$, provided $m$ is less than $n$.

(22) Prove that the series $\cos^m a+\cos^m(a+\phi)+$ etc. to $n$ terms, where $n\phi=2\pi$, is independent of $a$ if $m$ is less than $n$.

EXAMPLE 5. *To sum* $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \ldots$ *to* n *terms.*

We have
$$\operatorname{cosec}\theta = \cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^n\theta.$$

Therefore, as in Art. 67,
$$S_n = \cot\tfrac{1}{2}\theta - \cot 2^{n-1}\theta.$$

EXAMPLE 6. *To sum* $\tan\theta + \frac{1}{2}\tan\frac{1}{2}\theta + \frac{1}{4}\tan\frac{1}{4}\theta +$ etc. *to* n *terms.*

We have
$$\tan\theta = \cot\theta - 2\cot 2\theta,$$
$$\tfrac{1}{2}\tan\tfrac{1}{2}\theta = \tfrac{1}{2}\cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\tfrac{1}{4}\tan\tfrac{1}{4}\theta = \tfrac{1}{4}\cot\tfrac{1}{4}\theta - \tfrac{1}{2}\cot\tfrac{1}{2}\theta, \text{ etc.}$$

Therefore
$$S_n = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta.$$

69. If the result of summation of such series is given, it is often easy from that result to discover the required transformation.

For example. The result of the summation in Example 1 has $\sin\frac{1}{2}\delta$ in the denominator. This suggests that $\sin\alpha . \sin\frac{1}{2}\delta$ may be transformed into two quantities which are of course $\frac{1}{2}\cos(\alpha - \frac{1}{2}\delta) - \frac{1}{2}\cos(\alpha + \frac{1}{2}\delta)$. Again, in Examples (5) and (6) the required transformation will be at once seen if we *put* $n = 1$ *in the answer.*

The student however is advised only to resort to this method of solution as a last resource.

## EXAMPLES. XXVII.

Sum the following series to $n$ terms.

(1) $\sec\theta . \sec 2\theta + \sec 2\theta . \sec 3\theta + \sec 3\theta . \sec 4\theta + \ldots\ldots$

(2) $\operatorname{cosec}\theta . \operatorname{cosec} 2\theta + \operatorname{cosec} 2\theta . \operatorname{cosec} 3\theta + \operatorname{cosec} 3\theta . \operatorname{cosec} 4\theta + \ldots\ldots$

(3) $\operatorname{cosec}\theta . \sec 2\theta - \sec 2\theta . \operatorname{cosec} 3\theta + \operatorname{cosec} 3\theta . \sec 4\theta -$ etc.

(4) $\dfrac{1}{\cos a + \cos 3a} + \dfrac{1}{\cos a + \cos 5a} + \dfrac{1}{\cos a + \cos 7a} +$ etc.

(5) $\dfrac{\sin a}{\cos a + \cos 2a} + \dfrac{\sin 2a}{\cos a + \cos 4a} + \dfrac{\sin 3a}{\cos a + \cos 6a} +$ etc.

(6) $\dfrac{\cos a}{\cos a - \cos 2a} + \dfrac{\cos 2a}{\cos a - \cos 4a} + \dfrac{\cos 3a}{\cos a - \cos 6a} + \text{etc.}$

(7) $\sin 2\theta \,.\, \sin^2\theta + \tfrac{1}{2}\sin 4\theta \,.\, \sin^2 2\theta + \tfrac{1}{4}\sin 8\theta \,.\, \sin^2 4\theta + \ldots\ldots$

(8) $\sin 2\theta \,.\, \cos^2\theta - \tfrac{1}{2}\sin 4\theta \,.\, \cos^2 2\theta + \tfrac{1}{4}\sin 8\theta \,.\, \cos^2 4\theta - \ldots\ldots$

(9) $\sin\theta \,.\, \cos^2\dfrac{\theta}{2} - 2\sin\dfrac{\theta}{2}\cos^2\dfrac{\theta}{4} + 4\sin\dfrac{\theta}{4} \,.\, \cos^2\dfrac{\theta}{8} - \ldots\ldots$

(10) $\sec a \,.\, \sin 2a \,.\, \sec 3a + \sec 3a \,.\, \sin 4a \,.\, \sec 5a + \text{etc.}$

(11) $\dfrac{\sin 2a}{\sin a \,.\, \sin 3a} - \dfrac{\sin 4a}{\sin 3a \,.\, \sin 5a} + \text{etc.}$

(12) $\tan^{-1}\dfrac{1}{1+1\,.\,2} + \tan^{-1}\dfrac{1}{1+2\,.\,3} + \tan^{-1}\dfrac{1}{1+3\,.\,4} + \text{etc.}$

(13) $\tan^{-1}\dfrac{a}{1+1\,.\,2a^2} + \tan^{-1}\dfrac{a}{1+2\,.\,3a^2} + \tan^{-1}\dfrac{a}{1+3\,.\,4a^2} + \ldots$

(14) $\tan^{-1}\dfrac{3a^2}{1+1^2\,.\,2^2a^4} + \tan^{-1}\dfrac{5a^2}{1+2^2\,.\,3^2a^4} + \text{etc.}$

(15) $\tan\theta + 2\tan 2\theta + 4\tan 4\theta + \text{etc.}$

(16) $\tan a + \cot a + \tan 2a + \cot 2a + \tan 4a + \cot 4a + \text{etc.}$

(17) $\sin\theta \sin 3\theta + \sin 2\theta \sin 6\theta + \sin 2^2\theta \sin(2^2\,.\,3\theta) + \text{etc.}$

(18) $\sin\theta \,.\, \sin 3\theta + \sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2} + \sin\dfrac{\theta}{2^2} \,.\, \sin\dfrac{3\theta}{2^2} + \ldots\ldots$

(19) $\cot\theta \operatorname{cosec}\theta + 2\cot 2\theta \operatorname{cosec} 2\theta + 2^2\cot 2^2\theta \operatorname{cosec} 2^2\theta + \ldots$

(20) $\cot\theta \operatorname{cosec}\theta + \tfrac{1}{2}\cot\dfrac{\theta}{2}\operatorname{cosec}\dfrac{\theta}{2} + \dfrac{1}{2^2}\cot\dfrac{\theta}{2^2}\operatorname{cosec}\dfrac{\theta}{2^2} + \ldots$

(21) $\operatorname{cosec} a + \operatorname{cosec}\dfrac{a}{2} + \operatorname{cosec}\dfrac{a}{2^2} + \ldots\ldots$

(22) $\dfrac{1}{2}\sec a + \dfrac{1}{2^2}\sec a \sec 2a + \dfrac{1}{2^3}\sec a \sec 2a \sec 2^2 a + \text{etc.}$

(23) Deduce from Ex. 2 the sum of the series $\dfrac{1}{1\,.\,2} + \dfrac{1}{2\,.\,3} + \dfrac{1}{3\,.\,4}$ to $n$ terms.

(24) Deduce from Ex. 17 the sum of

$$1\,.\,3 + 2^2\,.\,1\,.\,3 + 2^4\,.\,1\,.\,3 + \ldots$$

to $n$ terms.

## SECOND METHOD.

70. When the sum of a series of the form

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \text{etc.}$$

is known, we can obtain the sums of two series of the forms

$$A_0 + A_1x\cos\theta + A_2x^2\cos 2\theta + A_3x^3\cos 3\theta + \text{etc.},$$

and $$A_1x\sin\theta + A_2x^2\sin 2\theta + A_3x^3\sin 3\theta + \text{etc.}$$

Let $C$ stand for the sum of the first series, and $S$ for the sum of the second series, then,

$$C+iS = A_0 + A_1x(\cos\theta + i\sin\theta) + A_2x^2(\cos 2\theta + i\sin 2\theta) + \text{etc.}$$

$$= A_0 + A_1xe^{i\theta} + A_2(xe^{i\theta})^2 + A_3(xe^{i\theta})^3 + \text{etc.}$$

The sum of the last series is known by hypothesis.

The result of the summation must then be expressed in the form $A+iB$; whence we have $C=A$ and $S=B$. [Art. 11.]

EXAMPLE 1. *Sum the series*

$$1 + x\cos\theta + x^2\cos 2\theta + \text{etc.} + x^{n-1}\cos(n-1)\theta.$$

Let $C_n$ be the sum of this series, and let

$$S_{n-1} = x\sin\theta + x^2\sin 2\theta + \ldots + x^{n-1}\sin(n-1)\theta.$$

Then $C_n + iS_{n-1} = 1 + xe^{i\theta} + x^2e^{2i\theta} + x^3e^{3i\theta} + \ldots x^{n-1}e^{(n-1)i\theta}$.

This is a series in G. P. $\therefore$ its sum $= \dfrac{1-x^ne^{in\theta}}{1-xe^{i\theta}}$.

Multiply the numerator and the denominator of this result each by $1-xe^{-i\theta}$, and we have

$$C_n+iS_{n-1} = \frac{(1-x^ne^{in\theta})(1-xe^{-i\theta})}{1-x(e^{i\theta}+e^{-i\theta})+x^2} = \frac{1-x^ne^{in\theta}-xe^{-i\theta}+x^{n+1}e^{i(n-1)\theta}}{1-2x\cos\theta+x^2}.$$

Therefore $C_n = \dfrac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}$. [Art. 11.]

Also $$S_{n-1} = \frac{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}.$$

EXAMPLE 2. *Sum the infinite series*

$$\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{\lfloor 2}\sin(\alpha+2\beta) + \frac{x^3}{\lfloor 3}\sin(\alpha+3\beta) + \ldots$$

Let $S$ stand for the sum of the series, and let

$$C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{\lfloor 2}\cos(\alpha+2\beta) + \ldots$$

Then $C + iS = e^{i\alpha} + xe^{i\alpha}e^{i\beta} + \frac{x^2}{\lfloor 2}e^{i\alpha}e^{i2\beta} + \ldots = e^{i\alpha}\{e^{xe^{i\beta}}\}$ [Art. 3.]

$$= e^{i\alpha}e^{x(\cos\beta + i\sin\beta)}$$

$$= (\cos\alpha + i\sin\alpha)e^{x\cos\beta}\{\cos(x\sin\beta) + i\sin(x\sin\beta)\}.$$

Therefore $S = e^{x\cos\beta}\{\sin\alpha\cos(x\sin\beta) + \cos\alpha\sin(x\sin\beta)\}$

$$= e^{x\cos\beta}\sin(\alpha + x\sin\beta).$$

## EXAMPLES. XXVIII.

Sum the following series to infinity.

(1) $\sin\alpha + x\sin 2\alpha + x^2\sin 3\alpha + \ldots$

(2) $\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots$

(3) $\sin\alpha + \cos\alpha\sin(\alpha+\beta) + \cos^2\alpha\sin(\alpha+2\beta) + \ldots$

(4) $\cos\alpha + \sin\alpha\cos(\alpha+\beta) + \sin^2\alpha\cos(\alpha+2\beta) + \ldots$

(5) $\sin\alpha + \frac{1}{\lfloor 2}\sin 2\alpha + \frac{1}{\lfloor 3}\sin 3\alpha + \ldots$

(6) $x\sin\alpha + \frac{x^2}{\lfloor 2}\sin 2\alpha + \frac{x^3}{\lfloor 3}\sin 3\alpha + \ldots$

(7) $1 - \frac{\cos\alpha}{\lfloor 1}\cos\beta + \frac{\cos^2\alpha}{\lfloor 2}\cos 2\beta - \frac{\cos^3\alpha}{\lfloor 3}\cos 3\beta + \ldots$

(8) $\sin\alpha - \frac{1}{\lfloor 1}\cos\alpha\sin(\alpha+\beta) + \frac{1}{\lfloor 2}\cos^2\alpha\sin(\alpha+2\beta) - \ldots$

(9) $\sin a - \frac{1}{2}\sin 2a + \frac{1}{3}\sin 3a - \ldots$

(10) $\cos a + \frac{1}{2}\cos 2a + \frac{1}{3}\cos 3a + \ldots$

(11) $\sin a \,.\, \cos\beta - \frac{1}{2}\sin^2 a \,.\, \cos 2\beta + \frac{1}{3}\sin^3 a \,.\, \cos 3\beta - \ldots$

(12) $\cos\theta \,.\, \cos\phi + \frac{1}{2}\cos^2\theta \,.\, \cos 2\phi + \frac{1}{3}\cos^3\theta \,.\, \cos 3\phi + \ldots$

Sum the series 13, 14, 17 to 20, to $n$ terms.

(13) $\cos a + x\cos(a+\beta) + x^2\cos(a+2\beta) + \ldots$

(14) $x\sin a - x^2\sin(a+\beta) + x^3\sin(a+2\beta) - \ldots$

(15) $1 + n\cos a + \dfrac{n(n-1)}{\lfloor 2}\cos 2a + \dfrac{n(n-1)(n-2)}{\lfloor 2}\cos 3a + \ldots$

to $(n+1)$ terms.

(16) $\sin a + nx\sin(a+\beta) + \dfrac{n(n-1)}{\lfloor 2}x^2\sin(a+2\beta) + \ldots$

to $(n+1)$ terms.

(17) $1 + \cos a \,.\, \cos\beta + \cos^2 a \,.\, \cos 2\beta + \cos^3 a \,.\, \cos 3\beta + \ldots$

(18) $\sin a + \sin a \,.\, \sin(a+\beta) + \sin^2 a \,.\, \sin(a+2\beta) + \ldots$

(19) $\sin a + 2\sin 2a + 3\sin 3a + \ldots$

(20) $1^2\cos a + 2^2\cos 2a + 3^2\cos 3a + \ldots$

*Sum the following series to infinity.

(21) $1 + \dfrac{p}{q}\cos a\sin\beta + \dfrac{p(p-q)}{\lfloor 2\, q^2}\cos^2 a\sin 2\beta + \dfrac{p(p-q)(p-2q)}{\lfloor 3\, q^3}\cos^3 a\sin 3\beta + \ldots$

(22) $\cos a - \frac{1}{3}\cos 3a + \frac{1}{5}\cos 5a - \ldots$

(23) $1 + \dfrac{1}{\lfloor 1}e^{\cos a}\cos(\sin a) + \dfrac{1}{\lfloor 2}e^{2\cos a}\cos(2\sin a) + \ldots$

(24) $e^{-x}\cos y - \frac{1}{3}e^{-3x}\cos 3y + \frac{1}{5}e^{-5x}\cos 5y - \ldots$

(25) $e^{x}\sin x - \frac{1}{2}e^{2x}\sin 2x + \frac{1}{3}e^{3x}\sin 3x - \ldots$

71. The expression $x^2 - 2x\cos\theta + 1$ is the product of the two factors $(1 - xe^{i\theta})(1 - xe^{-i\theta})$, and therefore an expression having $x^2 - 2x\cos\theta + 1$ for its denominator may often be expanded in ascending powers of $x$ by finding its equivalent partial fractions.

EXAMPLE I. *Expand* $\dfrac{2\cos\alpha - 2x\cos(\alpha-\beta)}{1 - 2x\cos\beta + x^2}$ *in ascending powers of* x.

$$\frac{2\cos\alpha - 2x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2} = \frac{e^{i\alpha}+e^{-i\alpha}-x(e^{i\alpha-i\beta}+e^{-i\alpha+\beta})}{(1-xe^{i\beta})(1-xe^{-i\beta})} \quad \text{[Art. 23.]}$$

$$= \frac{e^{i\alpha}}{1-xe^{i\beta}} + \frac{e^{-i\alpha}}{1-xe^{-i\beta}}$$

$$= e^{i\alpha}\{1 + xe^{i\beta} + x^2e^{2i\beta} + \ldots\} + e^{-i\alpha}\{1 + xe^{-i\beta} + x^2e^{-2i\beta} + \ldots\}$$

$$= e^{i\alpha} + e^{-i\alpha} + x(e^{i\alpha+i\beta} + e^{-i\alpha-i\beta}) + x^2(e^{2i\alpha+2i\beta} + e^{-2i\alpha-2i\beta}) + \ldots$$

$$= 2\cos\alpha + x\,2\cos(\alpha+\beta) + x^2\,2\cos 2(\alpha+\beta) + \ldots \quad \text{[Art. 23.]}$$

EXAMPLE II. *Expand* $\dfrac{1}{1 - 2x\cos\alpha + x^2}$ *in ascending powers of* x.

This expression may be written $(1 - xe^{i\alpha})^{-1}(1 - xe^{-i\alpha})^{-1}$

$$= \{1 + xe^{i\alpha} + x^2e^{2i\alpha} + x^3e^{3i\alpha} + \ldots\}\{1 + xe^{-i\alpha} + x^2e^{-2i\alpha} + x^3e^{-3i\alpha} + \ldots\}$$

$$= 1 + x^2 + x^4 + \ldots + (e^{i\alpha} + e^{-i\alpha})\{x + x^3 + x^5 + \ldots\}.$$

$$+ (e^{2i\alpha} + e^{-2i\alpha})\{x^2 + x^4 + x^6 + \ldots\} + \text{etc.}$$

$$= \frac{1}{1-x^2}\{1 + x\,2\cos\alpha + x^2\,2\cos 2\alpha + x^3\,2\cos 3\alpha + \ldots\}$$

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Writing $\dfrac{a}{b}$ for $x$ we have, if $a > b$

$$(a^2 - 2ab\cos\alpha + b^2)^{-1} = \frac{1}{a^2-b^2}\left\{1 + 2\frac{a}{b}\cos\alpha + 2\frac{a^2}{b^2}\cos 2\alpha + \ldots\right\}.$$

EXAMPLE III. *In any triangle* $c^2=a^2-2ab\cos C+b^2$; *let* a *be* > b,

*then* $$c^2=b^2\left(1-\frac{b}{a}e^{iC}\right)\left(1-\frac{b}{a}e^{-iC}\right), \qquad \text{[Art. 71.]}$$

$$\therefore\ 2\log c=2\log a+\log\left(1-\frac{b}{a}e^{iC}\right)+\log\left(1-\frac{b}{a}e^{-iC}\right)$$

$$=2\log a-\frac{b}{a}(e^{iC}+e^{-iC})-\tfrac{1}{2}\frac{b^2}{a^2}(e^{iC}+e^{-iC})-\text{etc.} \qquad \text{[Art. 4.]}$$

$$\therefore\quad \log c=\log a-\frac{b}{a}\cos C-\tfrac{1}{2}\frac{b^2}{a^2}\cos 2C-\tfrac{1}{3}\frac{b^2}{a^2}\cos 3C-\text{etc.} \qquad \text{[Art. 23.]}$$

This series may sometimes be made useful when $\frac{b}{a}$ is small.

72. The following example is important.

*Given* $\sin\theta=x\sin(\theta+\alpha)$, *expand* $\theta$ *in a series of ascending powers of* x.

Since $\sin\theta=x\sin(\theta+\alpha)$, $\therefore\ e^{i\theta}-e^{-i\theta}=x\,(e^{i\theta+i\alpha}-e^{-i\theta-i\alpha})$, [Art. 23.]

$$\therefore\quad e^{2i\theta}-1=x\{e^{i\alpha}e^{2i\theta}-e^{-i\alpha}\}.$$

$$\therefore\quad e^{2i\theta}=\frac{1-xe^{-i\alpha}}{1-xe^{i\alpha}}.$$

$$\therefore\quad \log e^{2i\theta}=\log(1-xe^{-i\alpha})-\log(1-xe^{i\alpha}).$$

$$\therefore\quad 2i\theta+2ir\pi=x\,(e^{i\alpha}-e^{-i\alpha})+\tfrac{1}{2}x^2\,(e^{2i\alpha}-e^{-2i\alpha})+\text{etc.} \qquad \text{[Art. 4.]}$$

$$\therefore\quad \theta+r\pi=x\sin\alpha+\tfrac{1}{2}x^2\sin 2\alpha+\tfrac{1}{3}x^3\sin 3\alpha+\ldots \qquad \text{[Art. 23, 28.]}$$

If in the above $x=-1$, then $\sin\theta=-\sin(\theta+\alpha)$, so that we may put $-2\theta$ for $\alpha$. Hence we obtain when $\theta$ is less than $\frac{1}{2}\pi$

$$\theta=\sin 2\theta-\tfrac{1}{2}\sin 4\theta+\tfrac{1}{3}\sin 6\theta-\text{etc.}$$

## EXAMPLES. XXIX.

(1) Expand $\dfrac{1}{1-2a\cos\phi+a^2}$ in a series of ascending powers of $a$; and prove that if $p_{n-1}$, $p_n$, $p_{n+1}$ be the coefficients of three consecutive terms $2p_n\cos\phi=p_{n-1}+p_{n+1}$.

Expand the following expressions in ascending powers of $x$.

(2) $\dfrac{\sin a}{1-2x\cos a+x^2}$.

(3) $\dfrac{1+x\cos\theta}{1+2x\cos\theta+x^2}$.

(4) $\log(1-2x\cos a+x^2)$.

(5) $\dfrac{\sin a-x\sin(a-\beta)}{1-2x\cos\beta+x^2}$.

(6) $e^{x\cos a}\cos(x\sin a)$.

(7) $e^{ax}\cos bx$.

(8) $e^{ax}\cos bx+e^{bx}\cos ax$.

(9) $e^{x\cos a}\sin(x\sin a)$.

(10) $e^{x\cos\beta}\cos(a+x\sin\beta)$.

(11) $e^{x\cos\beta}\sin(a+x\sin\beta)$.

(12) In any triangle $\sin A=\dfrac{a}{b}\sin(A+C)$, hence prove that

$$A=\frac{a}{b}\sin C+\frac{a^2}{2b^2}\sin 2C+\text{etc.}$$

(13) If $\tan\phi=n\tan\theta$, find a series for $\phi$ in terms of $\theta$.

(14) Prove that $\dfrac{\sec^n\theta}{2^n}=\frac{1}{2}\left\{\dfrac{e^{in\theta}}{(1+e^{2i\theta})^n}+\dfrac{e^{-in\theta}}{(1+e^{-2i\theta})^n}\right\}$, and expand $\sec^n\theta$ in cosines of multiples of $\theta$.

(15) Prove that $\cos na\cos^n a+i\sin na\cos^n a=\dfrac{1}{(1-i\tan a)^n}$ and expand $\cos na\cos^n a$ in ascending powers of $\tan a$.

(16) Sum to infinity the series

(i) $4+9\cos\theta+21\cos2\theta+51\cos3\theta+\text{etc.}$

(ii) $1+3x\sin\theta+11x^2\sin2\theta+43x^3\sin3\theta+\ldots$

## CHAPTER VI.

### Resolution of $\sin\theta$ and $\cos\theta$ into Factors.

73. *To prove* $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$

By Arts. 38, 52 or 62 we have when $n$ is a positive integer

$$x^{2n} - 2x^n \cos 2n\alpha + 1$$

$$= (x^2 - 2x\cos 2\alpha + 1)\left\{x^2 - 2x\cos\left(2\alpha + \frac{2\pi}{n}\right) + 1\right\} \ldots n \text{ factors}$$

in this result let $x = 1$, and let $2n\phi = \pi$, then

$$2(1-\cos 2n\alpha) = 2^n (1-\cos 2\alpha)\{1-\cos(2\alpha+4\phi)\}\ldots$$
$$\{1-\cos(2\alpha+\overline{n-1}\,4\phi)\}.$$

Now $1-\cos 2n\alpha = 2\sin^2 n\alpha$; hence taking the square root

$$\pm 2\sin n\alpha = 2^n \sin\alpha \,.\, \sin(\alpha+2\phi) \,.\, \sin(\alpha+4\phi)\ldots$$
$$\sin(\alpha+2n\phi-2\phi).$$

But $\sin(\alpha+2n\phi-2\phi) = \sin(\alpha+\pi-2\phi) = \sin(2\phi-\alpha)$,

hence, when $n$ is **odd**, we have

$$\pm 2\sin n\alpha = 2^n \sin\alpha \sin(2\phi+\alpha)\sin(2\phi-\alpha) \,.\, \sin(4\phi+\alpha)$$
$$\sin(4\phi-\alpha)\ldots\sin\{(n-1)\phi+\alpha\}\sin\{(n-1)\phi-\alpha\}.$$

But $\sin(2\phi+\alpha)\sin(2\phi-\alpha) = \sin^2 2\phi - \sin^2\alpha$. Hence

$$\pm 2\sin n\alpha = 2^n \sin\alpha\,(\sin^2 2\phi - \sin^2\alpha)\,(\sin^2 4\phi - \sin^2\alpha)\ldots$$

Next, divide both sides by $\sin a$, and let $a$ be diminished without limit, and we obtain

$$2n = 2^n \sin^2 2\phi \,.\, \sin^2 4\phi \sin^2 6\phi \ldots \sin^2 (n-1)\phi.$$

Divide the first of these last two results by the second, thus

$$\pm \sin na = n \sin a \left(1 - \frac{\sin^2 a}{\sin^2 2\phi}\right)\left(1 - \frac{\sin^2 a}{\sin^2 4\phi}\right) \ldots$$

Write $\theta$ for $na$, and let $n$ be increased while $a$ is diminished without limit, $\theta$ remaining unchanged: then since

$$\frac{\sin^2 a}{\sin^2 2\phi} = \frac{\sin^2 \dfrac{\theta}{n}}{\sin^2 \dfrac{\pi}{n}}, \quad \therefore \text{ the limit of } \frac{\sin^2 a}{\sin^2 2\phi} \text{ is } \frac{\theta^2}{\pi^2};$$

and the limit of $n \sin a$ = that of $n \dfrac{\theta}{n}$, *i.e.* $= \theta$;

proceeding to the limit we obtain

$$\pm \sin \theta = \theta \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \ldots$$

Now, when $\theta$ lies between 0 and $\pi$, $\sin \theta$ is positive and every factor on the right-hand side is positive; when $\theta$ lies between $\pi$ and $2\pi$, $\sin \theta$ is negative and one factor only on the right-hand side is negative; and so on. Therefore the upper sign must be taken in the above result instead of the ambiguity $\pm$: and the proposition is established.

*74. We can prove that the *upper* sign must be taken in each of the foregoing identities as follows:—

In the figure, let $ROP_0 = a$; produce $P_0O$ to $Q$, and divide the semicircumference $PQ$ into $n$ equal parts $P_0P_1$, $P_1P_2$, etc. Then since $n \,.\, 2\phi = \pi$, each of the angles $P_0OP_1$, $P_1OP_2 \ldots$ is equal to $2\phi$ and $ROP_1 = a + 2\phi$, $ROP_2 = a + 4\phi$, etc. Now consider the first ambiguity on page 74.

Fig.I.

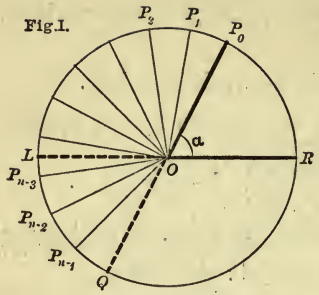
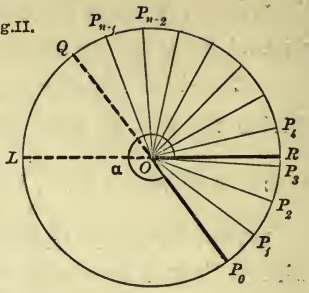



Fig.II.




I. **Let $\alpha$ be less than $\pi$ (Fig. I.).** Now, since $\sin ROP_{n-1}$ is *negative* when $P_{n-1}$ is *below* $ROL$, the product of sines on the right-hand side will be *positive* or *negative* according as the number of the points $P_{n-1}\,P_{n-2}\ldots$ which are below $ROL$, is *even* or *odd*. Let $r$ be that number. Then $\alpha$, which $= ROP_0 = LOQ$, is equal to $(r\,.\,2\phi + \lambda)$, where $\lambda$ is less than $2\phi$ [in Fig. I. $r = 3$ and $\lambda = LOP_{n-3}$]; hence $n\alpha = n\,(r2\phi + \lambda) = r\pi + n\lambda$, where $n\lambda$ is less than $\pi$. Therefore $\sin n\alpha$ is *positive* or *negative* according as $r$ is *even* or *odd*, that is, according as $\sin\alpha\,.\,\sin(\alpha + 2\phi)\sin(\alpha + 4\phi)\ldots$ is positive or negative.

II. **Let $\alpha$ lie between $\pi$ and $2\pi$ (Fig. II.).** Then $P_0$ is below $ROL_1$, and if there are *also* $r$ of the points $P_1, P_2\ldots$ below $ROL_1$, $(r+1)$ of the factors $\sin\alpha$, $\sin(\alpha+2\phi)\ldots$ are *negative*. And in this case $\alpha = 2\pi - P_0OR = 2\pi - (r\,.\,2\phi + \lambda)$ [in Fig. II. $r = 3$ and $P_3OR = \lambda$], $\therefore\ n\alpha = 2n\pi - r\pi - n\lambda$, where $n\lambda$ is less than $\pi$. Hence $\sin n\alpha = -\sin(r\pi + \lambda)$, and therefore is *negative* or *positive* according as $r$ is *even* or *odd*.

III. **Let $\alpha$ be greater than $2\pi$;** and let $\alpha = 2m\pi + \alpha'$ where $\alpha'$ is less than $2\pi$. And the proposition, being true for $\alpha'$ by I. and II., must also be true for $\alpha$.

75. In the identity

$$2 \sin n\alpha = 2^n \sin \alpha \,.\, \sin(\alpha + 2\phi) \sin(\alpha + 4\phi) \ldots \sin\{\alpha + (n-1)\,2\phi\}$$

write $\alpha + \phi$ for $\alpha$, then $n\alpha$ becomes $n\alpha + n\phi$, *i.e.* $n\alpha + \frac{1}{2}\pi$, and we have

$$2 \cos n\alpha = 2^n \sin(\alpha + \phi) \sin(\alpha + 3\phi) \sin(\alpha + 5\phi) \ldots \sin\{\alpha + (2n-1)\,\phi\}.$$

From this we can deduce as in Article 73 that, when $n$ is even,

$$2 \cos n\alpha = 2^n (\sin^2\phi - \sin^2\alpha)(\sin^2 3\phi - \sin^2\alpha) \ldots \{\sin^2(n-1)\,\phi - \sin^2\alpha\}.$$

Whence, writing $\theta$ for $n\alpha$ as before, we obtain

$$\cos\theta = \left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right)\left(1 - \frac{2^2\theta^2}{5^2\pi^2}\right)\ldots$$

[The ambiguity in the sign may be removed by the method either of Art. 73 or of Art. 74.]

76. Many particular identities may be obtained from the results proved in Arts. 73, 74. For example, in the identity

$$2 \cos n\alpha = 2^n \sin(\alpha + \phi) \sin(\alpha + 3\phi) \ldots \sin(\alpha + 2n\phi - \phi),$$

put $\alpha = 0$, and we have

$$1 = 2^{n-1} \sin\phi \,.\, \sin 3\phi \ldots \sin(2n-1)\,\phi, \text{ where } 2n\phi = \pi.$$

Again, in the identity

$$2 \sin n\alpha = 2^n \sin\alpha \sin(\alpha + 2\phi) \sin(\alpha + 4\phi) \ldots \sin(\alpha + 2n\phi - 2\phi),$$

let $\alpha$ be diminished without limit, and we have

$$2n = 2^n \sin 2\phi \sin 4\phi \ldots \sin(2n\phi - 2\phi), \text{ where } 2n\phi = \pi.$$

77. The two results

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

and

$$\cos\theta = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\ldots$$

could be proved very shortly as follows, if the following proposition were true.—*If a function of* $\theta$ (such as $\sin\theta$) *vanish for any value* $a$ *of* $\theta$, then $\theta - a$ *is a factor of that function of* $\theta$. This proposition is known to be true of any rational function of $\theta$, but it is not always true for infinite series like $\theta - \frac{\theta^3}{\lfloor\underline{3}} + \text{etc.}$;

[e.g. $e^{-\frac{1}{\theta^2}}$ vanishes when $\theta = 0$, but the series $1 - \frac{1}{\theta^2} + \frac{1}{\lfloor\underline{2}\theta^4} - \text{etc.}$, is not divisible by $\theta$. De Morgan's Differential Calculus, p. 176, Lond. 1842.]

*Assuming* the above proposition, it is clear that $\sin\theta$ is divisible by $\theta$, $\theta \pm \pi$, $\theta \pm 2\pi$, $\theta \pm 3\pi$, ... therefore

$$\sin\theta = A\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

Also $\sin\theta$ does not vanish for any other real values of $\theta$, nor for any imaginary values of $\theta$ [Example, page 29]. Hence $A$ does not contain $\theta$. And hence by diminishing $\theta$ indefinitely we obtain that the value of $A$ is 1.

Similarly, by assuming a corresponding proposition for $\cos\theta$, we have, since $\cos\theta$ vanishes when $\theta = \pm\frac{\pi}{2}$, or $\pm\frac{3\pi}{2}$, or $\pm\frac{5\pi}{2}$, ...

$$\cos\theta = A\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\ldots$$

and as before $A$ does not contain $\theta$; hence putting $\theta = 0$ we get $A = 1$.

*78. *To find the Factors of* $e^{2a} + e^{-2a} - 2\cos 2\theta$; *and of* sinh a, cosh a.

By Art. 63, $x^{2n} - 2x^n y^n \cos 2\alpha + y^{2n}$ has $n$ factors of the form

$$x^2 - 2xy\cos\frac{2r\pi + 2\theta}{n} + y^2.$$

Let $n$ be **odd.** The last factor is

$$x^2 - 2xy\cos\frac{2n\pi - 2\pi + 2\theta}{n} + y^2,$$

and this is equal to $x^2 - 2xy\cos\dfrac{2\pi - 2\theta}{n} + y^2$. The last factor but one is

$$x^2 - 2xy\cos\frac{2n\pi - 4\pi + 2\theta}{n} + y^2,$$

and this is equal to $x^2 - 2xy\cos\dfrac{4\pi - 2\theta}{n} + y^2$. And so on.

Hence $x^{2n} - 2x^n y^n \cos 2\theta + y^{2n}$

$$= \left(x^2 - 2xy\cos\frac{2\theta}{n} + y^2\right) \times \left(x^2 - 2xy\cos\frac{2\pi \pm 2\theta}{n} + y^2\right) \times$$

$$\ldots\left(x^2 - 2xy\cos\frac{2r\pi \pm 2\theta}{n} + y^2\right) \ldots 2n \text{ factors}$$

Where $x^2 - 2xy\cos\dfrac{2\pi \pm 2\theta}{n} + y^2$ stands for the two factors

$$\left(x^2 - 2xy\cos\frac{2\pi + 2\theta}{n} + y^2\right)\left(x^2 - 2xy\cos\frac{2\pi - 2\theta}{n} + y^2\right).$$

Now write $1 + \dfrac{a}{n}$ for $x$, and $1 - \dfrac{a}{n}$ for $y$.

The general form of factors on the right hand is

$$\left(1 + \frac{a}{n}\right)^2 - 2\left(1 - \frac{a^2}{n^2}\right)\cos\frac{2r\pi \pm 2\theta}{n} + \left(1 - \frac{a}{n}\right)^2,$$

that is, $$2\left(1+\frac{a^2}{n^2}\right)-2\left(1-\frac{a^2}{n^2}\right)\cos\frac{2r\pi\pm 2\theta}{n},$$

that is, $$4\sin^2\frac{r\pi\pm\theta}{n}\left\{1-\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}\right\}.$$

In the resulting equation put $a=0$, and we have

$$4\sin^2\theta = 4\sin^2\frac{\theta}{n}\,4\sin^2\frac{\pi\pm\theta}{n}\,4\sin^2\frac{4\pi\pm\theta}{n}\ldots$$

Using this result to simplify the right-hand side, we have

$$\left(1+\frac{a}{n}\right)^{2n}-2\left(1-\frac{a^2}{n^2}\right)^n\cos 2\theta+\left(1-\frac{a}{n}\right)^{2n}$$

$$=4\sin^2\theta\left(1+\frac{a^2}{n^2}\cot^2\frac{\theta}{n}\right)\ldots\left\{1+\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}\right\}\ldots$$

In this let $n$ be increased without limit, then

$\left(1+\frac{a}{n}\right)^{2n}$ *i.e.* $\left\{\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right\}^{2a}$ becomes $e^{2a}$. [Art. 2.]

$\left(1-\frac{a}{n}\right)^{2n}$ *i.e.* $\left\{\left(1-\frac{a}{n}\right)^{-\frac{n}{a}}\right\}^{-2a}$ becomes $e^{-2a}$.

$\left(1-\frac{n^2}{a^2}\right)^n$ *i.e.* $\left(1-\frac{n}{a}\right)^n\left(1+\frac{n}{a}\right)^n$ becomes $e^a\times e^{-a}$ *i.e.* 1.

$\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}$ becomes $\frac{n^2}{(r\pi\pm\theta)^2}$. Hence

$$e^{2a}+e^{-2a}-2\cos 2\theta = 4\sin^2\theta\left\{1+\frac{a^2}{\theta^2}\right\}\left\{1+\frac{a^2}{(\pi\pm\theta)^2}\right\}$$

$$\left\{1+\frac{a^2}{(2\pi\pm\theta)^2}\right\}\ldots\left\{1+\frac{a^2}{(r\pi\pm\theta)^2}\right\}\ldots$$

*79. In this result put $\theta=0$; then the limit of $\sin^2\theta\left\{1+\frac{a^2}{\theta^2}\right\}$ is $a^2$; and we obtain

$$\left(\frac{e^a-e^{-a}}{2}\right)^2 = a^2\left(1+\frac{a^2}{\pi^2}\right)^2\left(1+\frac{a^2}{2^2\pi^2}\right)^2\ldots\left(1+\frac{a^2}{r^2\pi^2}\right)^2\ldots$$

Next put $2\theta=\pi$, and we have

$$\left(\frac{e^a+e^{-a}}{2}\right)^2 = \left(1+\frac{2^2a^2}{\pi^2}\right)^2\left(1+\frac{2^2a^2}{3^2\pi^2}\right)^2\ldots\left(1+\frac{2^2a^2}{r^2\pi^2}\right)^2\ldots$$

In taking the square root, since $e^a-e^{-a}$ has always the same sign as $a$ [Art. 2], and $e^a+e^{-a}$ is always positive, we must take the same sign for each side in each result. Hence

$$\sinh a = a\left(1+\frac{a^2}{\pi^2}\right)\left(1+\frac{a^2}{2^2\pi^2}\right)\left(1+\frac{a^2}{3^2\pi^2}\right)\ldots$$

$$\cosh a = \left(1+\frac{2^2a^2}{\pi^2}\right)\left(1+\frac{2^2a^2}{3^2\pi^2}\right)\left(1+\frac{2^2a^2}{5^2\pi^2}\right)\ldots$$

80. Since $$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc.}$$

and $$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots$$

we obtain the Algebraical identity

$$1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\text{etc.} = \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots \quad \text{(I.)}$$

Similarly, from the two expressions equivalent to $\cos\theta$, we obtain

$$1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\text{etc.} = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\ldots \text{(II.)}$$

I. and II. are identities, and are true for all values of $\theta$. Therefore they are true if we write $\phi$ for $\theta^2$; also they are true when in the resulting identities for $\phi$, we write $-\phi$ for $\phi$.

In the resulting identities we may write $a^2$ for $\phi$, then

$$1+\frac{a^2}{\lfloor 3}+\frac{a^4}{\lfloor 5}+\ldots=\left(1+\frac{a^2}{\pi^2}\right)\left(1+\frac{a^2}{2^2\pi^2}\right)\left(1+\frac{a^2}{3^2\pi^2}\right)\ldots$$

$$1+\frac{a^2}{\lfloor 2}+\frac{a^4}{\lfloor 4}+\ldots=\left(1+\frac{2^2a^2}{\pi^2}\right)\left(1+\frac{2^2a^2}{3^2\pi^2}\right)\left(1+\frac{2^2a^2}{5^2\pi^2}\right)\ldots$$

By the above artifice the results of Art. 77 and of Arts. 73, 74 may be deduced the one from the other without the introduction of $\sqrt{(-1)}$.

81. Many results may be obtained from the identities

$$1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\text{etc.}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots$$

$$1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\text{etc.}=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\ldots$$

EXAMPLE 1. *Prove that* $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$.

From first of the above identities we have

$$\log\left(1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\text{etc.}\right)=\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\text{etc.}$$

Expanding each of these logarithms by Art. 4, we have

$$\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\ldots\right)+\tfrac{1}{2}\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\ldots\right)^2+\text{etc.}$$

$$=\left(\frac{\theta^2}{\pi^2}+\tfrac{1}{2}\frac{\theta^4}{\pi^4}+\text{etc.}\right)+\left(\frac{\theta^2}{2^2\pi^2}+\tfrac{1}{2}\frac{\theta^4}{2^4\pi^4}+\text{etc.}\right)$$

$$+\left(\frac{\theta^2}{3^2\pi^2}+\tfrac{1}{2}\cdot\frac{\theta^4}{3^4\pi^4}+\text{etc.}\right)+\text{etc.}$$

$$=\frac{\theta^2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)+\frac{\theta^4}{2\pi^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right).$$

In this identity we may equate the coefficients of the various powers of $\theta^2$. Hence

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{1\,.\,2\,.\,3}.$$

*EXAMPLE 2. *To prove* $\cot\theta=\frac{1}{\theta}-\frac{2\theta}{\pi^2-\theta^2}-\frac{2\theta}{2^2\pi^2-\theta^2}-\frac{2\theta}{3^2\pi^2-\theta^2}-\dots$

By Art. 73, $\log\sin\theta=\log\theta+\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\dots$ I.

The required result may be obtained by writing $\theta+h$ for $\theta$ in this identity, expanding each term in ascending powers of $\theta$, and then equating the coefficient of $h$ on each side. Now

$$\log\sin(\theta+h)=\log\sin\theta+h\cot\theta-\tfrac{1}{2}h^2\operatorname{cosec}^2\theta-\text{etc. as in Art. 101}$$

$$\log(\theta+h)=\log\left\{\theta\left(1+\frac{h}{\theta}\right)\right\}=\log\theta+\frac{h}{\theta}+\tfrac{1}{2}\frac{h^2}{\theta^2}+\dots \qquad \text{[Art. 4.]}$$

$$\log\left\{1-\frac{(\theta+h)^2}{r^2\pi^2}\right\}=\log\left\{1-\frac{\theta^2}{r^2\pi^2}-\frac{2h\theta}{r^2\pi^2}-\frac{h^2}{r^2\pi^2}\right\}$$

$$=\log\left\{\left(1-\frac{\theta^2}{r^2\pi^2}\right)\left(1-\frac{2h\theta}{r^2\pi^2-\theta^2}-\frac{h^2}{r^2\pi^2-\theta^2}\right)\right\}$$

$$=\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)+\log\left(1-\frac{2h\theta}{r^2\pi^2-\theta^2}-\frac{h^2}{r^2\pi^2-\theta^2}\right)$$

$$=\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)-\frac{2h\theta}{r^2\pi^2-\theta^2}-h^2\frac{r^2\pi^2+\theta^2}{(r^2\pi^2-\theta^2)^2}-\text{etc.}\quad\text{[Art. 4.]}$$

Hence, making these substitutions in

$$\log\sin(\theta+h)=\log(\theta+h)+\log\left\{1-\frac{(\theta+h)^2}{\pi^2}\right\}+\text{etc.},$$

and equating the coefficients of $h$, we obtain the required result.

## EXAMPLES. XXX.

Prove the following statements:

(1) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots=\frac{1}{8}\pi^2$.

(2) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots=\frac{1}{90}\pi^4$.

(3) $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\dots=\frac{1}{96}\pi^4$.

(4) The sum of the products of the squares of the reciprocals of every pair of positive integers is $\frac{1}{120}\pi^4$.

(5) $\frac{\pi}{2}=\frac{2^2}{1.3}\cdot\frac{4^2}{3.5}\cdot\frac{6^2}{5.7}\cdots$

(6) When $n$ is even

(i) $2^{\frac{n-1}{2}} \sin\frac{\pi}{2n} \cdot \sin\frac{3\pi}{2n} \ldots \sin\frac{n-1}{2n}\pi = 1.$

(ii) $2^{\frac{n-1}{2}} \cos\frac{\pi}{n} \cdot \cos\frac{2\pi}{n} \ldots \cos\frac{n-4}{2n}\pi \cdot \cos\frac{n-2}{2n}\pi = \sqrt{n}.$

(7) $$\frac{\sin(\theta-a)}{\sin a} = \left(1-\frac{\theta}{a}\right)\left(1+\frac{\theta}{\pi-a}\right)\left(1-\frac{\theta}{\pi+a}\right)\left(1+\frac{\theta}{2\pi-a}\right) \times \left(1-\frac{\theta}{2\pi+a}\right)\left(1+\frac{\theta}{3\pi-a}\right)\left(1-\frac{\theta}{3\pi+a}\right)\ldots\ldots$$

(8) $$\frac{\sin(a-\theta)}{\sin a} = \left(1+\frac{\theta}{a}\right)\left(1-\frac{\theta}{\pi-a}\right)\left(1+\frac{\theta}{\pi+a}\right)\left(1-\frac{\theta}{2\pi-a}\right)\ldots.$$

(9) $$\frac{\cos(a-\theta)}{\cos a} = \left(1+\frac{2\theta}{\pi-2a}\right)\left(1-\frac{2\theta}{\pi+2a}\right)\left(1+\frac{2\theta}{3\pi-2a}\right) \times \left(1-\frac{2\theta}{3\pi+2a}\right)\left(1+\frac{2\theta}{5\pi-2a}\right)\left(1-\frac{2\theta}{5\pi+2a}\right)\ldots\ldots$$

(10) $$\frac{\cos(a+\theta)}{\cos a} = \left(1-\frac{2\theta}{\pi-2a}\right)\left(1+\frac{2\theta}{\pi+2a}\right)\left(1-\frac{2\theta}{3\pi-2a}\right)\ldots\ldots$$

(11) $$\frac{\cos\theta+\cos a}{1+\cos a} = \frac{\cos\frac{1}{2}(a-\theta)\cos\frac{1}{2}(a+\theta)}{\cos\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \left\{1-\frac{\theta^2}{(\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(5\pi\pm a)^2}\right\}\ldots\ldots$$

where $1-\dfrac{\theta^2}{(\pi\pm a)^2}$ stands for $\left\{1-\dfrac{\theta^2}{(\pi+a)^2}\right\}\left\{1-\dfrac{\theta^2}{(\pi-a)^2}\right\}$.

(12) $$\frac{\cos\theta-\cos a}{1-\cos a} = \left(1-\frac{\theta^2}{a^2}\right)\left\{1-\frac{\theta^2}{(\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(2\pi\pm a)^2}\right\}\ldots\ldots$$

(13) $$\frac{\sin\theta+\sin a}{\sin a} = \left(1+\frac{\theta}{a}\right)\left(1+\frac{\theta}{\pi-a}\right)\left(1-\frac{\theta}{\pi+a}\right)\left(1+\frac{\theta}{2\pi+a}\right)\left(1-\frac{\theta}{2\pi-a}\right)\ldots\ldots$$

(14) $\dfrac{\sin a-\sin\theta}{\sin a}=\left(1-\dfrac{\theta}{a}\right)\left(1-\dfrac{\theta}{\pi-a}\right)\left(1+\dfrac{\theta}{\pi+a}\right)\left(1-\dfrac{\theta}{2\pi+a}\right)\ldots.$

(15) From the result of Ex. (11) deduce the factors of

$$\cosh x+\cos a.$$

(16) From the result of Ex. (12) deduce the factors of

$$\cosh x-\cos a.$$

(17) From the result of Ex. (7) prove that

$$\cot a=\frac{1}{a}-\frac{2a}{\pi^2-a^2}-\frac{2a}{2^2\pi^2-a^2}-\frac{2a}{3^2\pi^2-a^2}-\ldots\ldots$$

(18) $\tan x=\dfrac{2}{\pi-2x}-\dfrac{2}{\pi+2x}+\dfrac{2}{3\pi-2x}-\dfrac{2}{3\pi+2x}+\text{etc.}$

(19) $\dfrac{1}{\sin a}=\dfrac{1}{a}+\dfrac{2a}{\pi^2-a^2}-\dfrac{2a}{2^2\pi^2-a^2}+\dfrac{2a}{3^2\pi^2-a^2}-\ldots$

(20) $\dfrac{1}{4\cos a}=\dfrac{\pi}{\pi^2-2^2a}-\dfrac{3\pi}{3^2\pi^2-2^2a^2}+\dfrac{5\pi}{5^2\pi^2-2^2a^2}-\ldots$

(21) Since $e^{2\iota x}+e^{-2\iota x}-2\cos 2a=2\cos 2\iota x-2\cos 2a$

$$=4\sin(a+\iota x)\sin(a-\iota x),$$

deduce from the factors of $\sin\theta$ those of $\cosh 2x-\cos 2a$.

(22) $\dfrac{1}{x^2+1^2}+\dfrac{1}{x^2+3^2}+\dfrac{1}{x^2+5^2}+\text{etc.}=\dfrac{\pi}{4x}\dfrac{e^{\frac{\pi x}{2}}-e^{-\frac{\pi x}{2}}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}.$

(23) $\dfrac{1}{1^2+x^2}+\dfrac{1}{2^2+x^2}+\dfrac{1}{3^2+x^2}+\text{etc.}=\dfrac{\pi}{2x}\cdot\dfrac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}}-\dfrac{1}{2x^2}.$

(24) $\left(1+\dfrac{2}{1+1^2}+\dfrac{2}{1+2^2}+\dfrac{2}{1+3^2}+\ldots\right)$

$$\times\left(\frac{1}{4+1^2}+\frac{1}{4+3^2}+\frac{1}{4+5^2}+\ldots\right)=\frac{\pi^2}{8}.$$

(25) $\operatorname{cosec}^2\theta=-\dfrac{1}{\theta^2}+\tfrac{1}{2}\dfrac{\pi^2+\theta^2}{(\pi^2-\theta^2)^2}+\tfrac{1}{2}\dfrac{2^2\pi^2+\theta^2}{(2^2\pi^2-\theta^2)^2}$

$$+\tfrac{1}{2}\frac{3^2\pi^2+\theta^2}{(3^2\pi^2-\theta^2)^2}+\ldots$$

## *CHAPTER VII.

### On the use of $\sqrt{(-1)}$.

82. We propose now to make a few remarks on the use of $\sqrt{(-1)}$.

$\sqrt{(-1)}$ has been defined [Art. 11] as a symbolical expression whose square is $(-1)$, which is subject to all the laws of Algebra.

Thus the factor $\sqrt{(-1)}$ must not be considered a *quantity* but a *symbol of operation.*

In the same way the factor $(-1)$ is to be considered not a *quantity* but a *symbol of operation* [cf. E. Chapter VIII.].

Now the laws of Symbolical Algebra are identical with those of Arithmetical Algebra. Consequently any *general* result in Arithmetical Algebra must be considered true for all values of the symbols involved, provided those values are *subject to the laws of Algebra.*

Conversely, when any result in Symbolical Algebra is capable of an Arithmetical Interpretation, we ought to obtain a result which is *Arithmetically true.*

This important principle is called the Principle of the **Permanence of Equivalent Forms.**

EXAMPLE 1. In Euler's proof of the Binomial Theorem for any value of the index, having defined $f(m)$ we say that since

$$f(m) \times f(n) = f(m+n)$$

when $m$ and $n$ are positive integers, therefore $f(m) \times f(n) = f(m+n)$ when $m$ and $n$ have any values, *provided those values obey the laws of multiplication.*

EXAMPLE 2. In Art. 3 we have the following general theorem:

$$a^x = 1 + \frac{x \log_e a}{1} + \frac{x^2 (\log_e a)^2}{1.2} + \frac{x^3 (\log_e a)^3}{1.2.3} + \ldots$$

which is proved for all arithmetical values of $x$ and $a$.

Now since $i$, that is $\sqrt{(-1)}$, is a symbol which obeys all the laws of Algebra, we may put $\cos a + i \sin a$ for $a$ in the formula; as we have done in Art. 23.

83. We have in the proposition of Art. 23 a means of testing the truth of this principle. For, from the result of Art. 23 we obtain many important results. Now we shall find, that we are unable by any legitimate process of Algebra to get any result from that of Art. 23 which can be proved false by some other means; in fact every result obtained from Art. 23 which can be tested by some other process will be found to be true.

EXAMPLE. Take the results of Art. 26 and 32; in Art. 41 an independent proof is given of the result of Art. 26. Gregory's series may be proved by the method of Indeterminate Coefficients; and the student will find that Gregory's series and all other series of a similar kind, whose Trigonometrical Proofs depend on the use of $\sqrt{(-1)}$ and therefore on the principle of the Permanence of Equivalent Forms, can be verified by the Differential Calculus.

84. To those who feel a difficulty in accepting proofs depending on the use of $\sqrt{(-1)}$, we recommend a careful comparison of Art. 47 with Art. 37, and of Art. 40 with Art. 19. The proofs in each case are virtually the same, but in the one case each step is capable of an arithmetical interpretation, in the other the work is *abbreviated* by the aid of the symbol $\sqrt{(-1)}$.

85. A complete Geometrical interpretation has been given to the symbol $\sqrt{(-1)}$ of which the following is a short sketch. The student is referred to Professor De Morgan's Trigonometry and Double Algebra for further information on the whole subject.

Take any origin $O$ and any initial line $OR$. Let the letters $a$, $b$, etc. be the measures of the *lengths* of lines expressed in some fixed unit.

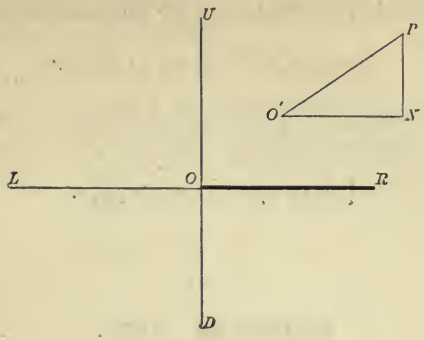
Then we make the following **Convention.**

$+a$ is to represent a line of length $a$ drawn parallel to the direction from $O$ to $R$.

$\sqrt{(-1)}\,a$, or $\iota a$, is to represent a line of length $a$ drawn parallel to the direction from $O$ to $U$ *at right angles* to $OR$.

Then $\sqrt{(-1)}\,\{\sqrt{(-1)}\,a\}$ will represent a line drawn at right angles to $\sqrt{(-1)}\,a$ of length $a$. That is $-a$ will represent a line drawn parallel to the direction from $O$ to $L$ (opposite to $OR$). This is in accordance with the convention already laid down in elementary Algebra [E. 122].

86. In accordance with the above Convention $a+\sqrt{(-1)}b$ will indicate the 'sum' of two lines, one $a$ units long, drawn parallel to $OR$, and the other $b$ units long, drawn parallel to $OB$.




Let $O'N = +a$ and $NP = \sqrt{(-1)}\, b$.

Then we may consider $O'P$ equivalent to $a + \sqrt{(-1)}\, b$.

For $O'N + NP$ means, 'go from $O'$ to $N$ and from $N$ to $P$;' $O'P$ means 'go from $O'$ to $P$.' [E. Chapter VIII.]

Now $O'P = \sqrt{O'N^2 + PN^2} = \sqrt{a^2 + b^2}$.

Thus the **magnitude** of $O'P$ is what is called in Algebra the **modulus** of the expression $a + \sqrt{(-1)}\, b$.

87. Again, let $a + \iota b = r(\cos\theta + \iota \sin\theta)$,

then
$$r = \sqrt{a^2 + b^2} = O'P,$$

$$\tan\theta = \frac{b}{a} = \tan NO'P.$$

Thus $(\cos\theta + \iota \sin\theta)\, r$ (where $r$ is the number of units of length in $O'P$ and $\theta = NO'P$) *represents $O'P$ in direction and magnitude.*

Hence $\cos\theta + \iota \sin\theta$ is a symbol of operation, and interpreted geometrically means 'turn the line operated on through an angle $\theta$.'

Hence we have immediately De Moivre's Theorem. For

$$(\cos\beta + \iota \sin\beta)(\cos\alpha + \iota \sin\alpha) r$$

means first turn the line of length $r$ through an angle $\alpha$ and then on through an angle $\beta$. This is clearly equivalent to

$$\{\cos(\alpha+\beta) + \iota \sin(\alpha+\beta)\} r.$$

## EXAMPLES. XXXI.

(1) Prove that the two expressions

$$r(\cos\alpha + \iota \sin\alpha) + r'(\cos\beta + \iota \sin\beta)$$

and

$$\rho(\cos\theta + \iota \sin\theta)$$

where $\rho^2 = r^2 + r'^2 - 2rr' \cos(\alpha-\beta)$ and $\tan\theta = \dfrac{r\cos\alpha + r'\cos\beta}{r\sin\alpha + r'\sin\beta}$

are equivalent algebraically and geometrically.

(2) Prove that the factor $e^{2ir\pi}$ where $r$ is a whole number, is a factor which does not alter the quantity multiplied.

## CHAPTER VIII.

### THE RULE OF PROPORTIONAL DIFFERENCES, OTHERWISE CALLED THE THEORY OF PROPORTIONAL PARTS.

88. THE logarithms in this Chapter are Common Logarithms.

In the Elementary Trigonometry a Rule called the Rule of Proportional Differences was given, and it was shown that, assuming the Rule to be *practically* true, we are enabled to use Tables of a more moderate size than would otherwise be necessary. [Cf. E. 218—222.]

The RULE is as follows. The differences between three numbers are proportional to the corresponding differences between the logarithms of those numbers, provided the differences between the numbers are small compared with the numbers.

In this Chapter we shall prove that this Rule of Proportional Differences is practically true as applied to the Table of Logarithms of Numbers, and that it is also practically true *in general* as applied to Tables of Trigonometrical Ratios and their Logarithms.

89. The Rule as applied to the Tables of Logarithms may be stated thus:

Let $n$ be any number greater than 10000 and less than 100000; let $d$ be any number not greater than unity; then as far as seven places of decimals the following proportion is true:

$$\frac{\log(n+d)-\log n}{\log(n+1)-\log n}=\frac{d}{1}.$$

90. *To prove the Rule for the Table of* **common logarithms.**

We have

$$\log(n+d)=\log n\left(1+\frac{d}{n}\right)=\log n+\log\left(1+\frac{d}{n}\right)$$

$$=\log n+\mu\left\{\frac{d}{n}-\frac{1}{2}\frac{d^2}{n^2}+\frac{1}{3}\frac{d^3}{n^3}-\ldots.\right\}. \quad [\text{Art. 4.}]$$

Let $n$ be not less than 10000 and $d$ not greater than 1; also $\mu$ the modulus [Art. 7] is ·4342945.... Hence $\mu$ is less than ·5, $\frac{d}{n}$ is not greater than ·0001. Therefore $\frac{1}{2}\frac{\mu d^2}{n^2}$ is not greater than $\frac{1}{4}(\cdot0001)^2$, *i.e.* not greater than ·0000000025; $\frac{\mu}{3}\frac{d^3}{n^3}$ is much less than this.

Hence *at least as far as seven decimal places*

$$\log(n+d)-\log n=\frac{\mu d}{n}.$$

Similarly $\log(n+1) - \log n = \frac{\mu}{n}$.

Therefore $\frac{\log(n+d) - \log n}{\log(n+1) - \log n} = \frac{d}{1}$;

which proves the Rule.

91. *To prove the Rule for the Table of natural* sines.

That is, *To prove that if* $\alpha$ *be any angle, and* $\delta$, $\delta'$ *angles not greater than* 1′, *then*

$$\frac{\sin(\alpha+\delta) - \sin\alpha}{\sin(\alpha+\delta') - \sin\alpha} = \frac{\delta}{\delta'}$$

*as far as seven decimal places.*

We have $\sin(\alpha+\delta) = \sin\alpha\cos\delta + \cos\alpha\sin\delta$

$= \sin\alpha\{1 - \frac{1}{2}\delta^2 + \ldots\} + \cos\alpha\{\delta - \frac{1}{6}\delta^3 + \ldots\}$ [Art. 41.]

$= \sin\alpha + \delta\cos\alpha - \frac{1}{2}\delta^2\sin\alpha - \frac{1}{6}\delta^3\cos\alpha + \ldots$.

$\delta$ is here the circular measure of an angle not greater than 1′, $\therefore$ $\delta$ is not greater than ·0003 [E. Ex. x. 17]; $\therefore$ $\frac{1}{2}\delta^2$ is not greater than ·00000005 and $\sin\alpha$ is not greater than 1. Hence, as far as seven places of decimals in the value of the sines,

$$\sin(\alpha+\delta) - \sin\alpha = \delta\cos\alpha.$$

Similarly $\sin(\alpha+\delta') - \sin\alpha = \delta'\cos\alpha$.

Therefore $\frac{\sin(\alpha+\delta) - \sin\alpha}{\sin(\alpha+\delta') - \sin\alpha} = \frac{\delta}{\delta'} = \frac{n}{n'}$,

where $n$ and $n'$ are the numbers of seconds in the angles $\delta$ and $\delta'$ respectively.

Thus the rule of proportional difference is true as applied to the Table of natural sines.

92. It must be observed however, that when $\alpha$ is nearly $\frac{1}{2}\pi$, $\cos\alpha$ is very small, and $\sin\alpha$ approaches unity; so that when $\alpha$ is nearly $\frac{1}{2}\pi$, $\delta\cos\alpha$ is comparable with

$$\tfrac{1}{2}\delta^2\sin\alpha.$$

Hence, when $\alpha$ is within a few minutes of $90^\circ$ we cannot neglect $-\frac{1}{2}\delta^2\sin\alpha$ *in comparison with* $\delta\cos\alpha$. Therefore in this case we must say that

$$\sin(\alpha+\delta)-\sin\alpha=\delta\cos\alpha-\tfrac{1}{2}\delta^2\sin\alpha.$$

Hence, the differences between the sines of two angles which are each nearly $90^\circ$, are *not* approximately proportional to the difference between the angles. The differences are then said to be **irregular.**

It must be noticed, however, that we have proved that $\frac{1}{2}\delta^2\sin\alpha$ is less than ·00000005, and as *we are neglecting figures after seven places* of decimals the term $-\frac{1}{2}\delta^2\sin\alpha$ does not affect the result.

A quantity whose measure when expressed in terms of the unit under consideration is less than ·0000001 is said to be **insensible.**

Consequently, in the case of sines, when the differences become *irregular* they are at the same time *insensible.* This insensibility gives rise to a serious practical difficulty. See Arts. 106, 119.

93. *The case of the Natural* **cosines.**

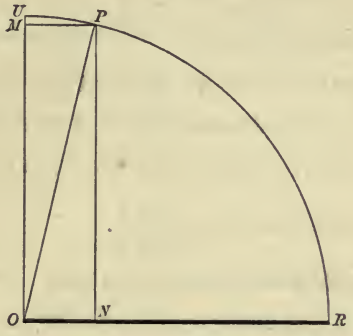
Since $\cos(\alpha+\delta)=\cos\alpha\cos\delta-\sin\alpha\sin\delta$

$$=\cos\alpha-\delta\sin\alpha-\tfrac{1}{2}\delta^2\cos\alpha+\tfrac{1}{6}\delta^3\sin\alpha+\ldots,$$

we may prove as in Art. 91 that the rule of proportional differences is practically applicable to the cosines of angles when the differences between the angles are less than one minute.

Also, we may prove as in Art. 92 that when $\alpha$ is nearly zero the differences between the cosines are **irregular**, but **insensible.**

94. The above results may be explained geometrically thus:




Let *RPU* be the quadrant of a circle, radius unity. Let *ROP* be an angle nearly 90°; draw *PN*, *PM* perpendiculars to *OR* and *OU*.

Then the arc *UP* and the line *PM* *approach to coincidence* as the angle *POU* is diminished. In other words the path of *P* differs very little from the line *PM*. Hence, when *ROP* is nearly a right angle the differences in *PN*, as the angle approaches a right angle, are small when compared with the differences in the arc *RP*. Hence it is said that the differences in $\sin\theta$ when $\theta$ is nearly $\frac{1}{2}\pi$ are insensible. The irregularity in the differences is caused by their smallness and consequently it is of little importance.

Also since the measure of *PN* is the cosine of the angle *UOP*, we can see from the same figure the cause of the insensibility of difference in the cosine of a *small* angle.

95. The case of the **tangent** may be discussed as follows:

$$\tan(\theta+\delta) = \frac{\sin(\theta+\delta)}{\cos(\theta+\delta)} = \frac{\sin\theta + \delta\cos\theta - \frac{1}{2}\delta^2\sin\theta - \text{etc.}}{\cos\theta - \delta\sin\theta - \frac{1}{2}\delta^2\cos\theta + \text{etc.}}$$

$$= \frac{\sin\theta\,(1 + \delta\cot\theta - \frac{1}{2}\delta^2 - \text{etc.})}{\cos\theta\,(1 - \delta\tan\theta - \frac{1}{2}\delta^2 + \text{etc.})}$$

hence, neglecting higher powers of $\delta$ than the second, we have

$$\tan(\theta+\delta) = \tan\theta\,(1 + \delta\cot\theta - \tfrac{1}{2}\delta^2)\{1 - (\delta\tan\theta + \tfrac{1}{2}\delta^2)\}^{-1}$$

$$= \tan\theta\,(1 + \delta\cot\theta - \tfrac{1}{2}\delta^2)(1 + \delta\tan\theta + \tfrac{1}{2}\delta^2 + \delta^2\tan^2\theta)$$

$$= \tan\theta\{1 + \delta\,(\cot\theta + \tan\theta) + \delta^2\,(1 + \tan^2\theta)\}$$

$$= \tan\theta + \delta\sec^2\theta + \delta^2\frac{\sin\theta}{\cos^3\theta}.$$

Hence, unless $\sin\theta\sec^3\theta$ is large, we have

$$\tan(\theta+\delta) - \tan\theta = \delta\sec^2\theta,$$

which proves the rule in this case.

96. Suppose that the Table of tangents is calculated for every minute. Then the largest value of $\delta$ (as in Art. 91) is ·0003 nearly. Hence the greatest value of $\delta^2\sin\theta\sec^3\theta$ is (·00000009) $\sin\theta\sec^3\theta$ nearly. So that when $\theta$ is greater than $\frac{1}{4}\pi$ we are liable to an error in the seventh place of decimals. Hence the Rule is not true for tables of tangents calculated for every minute, when the angle is between 45° and 90°.

97. It follows immediately from Art. 95, since the **cotangent** of an angle is equal to the tangent of its complement, that the Rule must not be used for a table of cotangents, calculated for every minute, when the angle lies between 0 and 45°.

98. The case of the **secant.**

$$\sec(\theta+\delta)=\frac{1}{\cos(\theta+\delta)}=\frac{1}{\cos\theta(1-\delta\tan\theta-\frac{1}{2}\delta^2+\ldots)}$$
$$=\sec\theta\{1+\delta\tan\theta+\delta^2(\tfrac{1}{2}+\tan^2\theta)+\ldots\}.$$

Hence, neglecting powers of $\delta$ above the second,

$$\sec(\theta+\delta)-\sec\theta=\delta\sin\theta\sec^2\theta+\delta^2(\tfrac{1}{2}+\tan^2\theta)\sec\theta.$$

Whence it may be shewn that the differences are *irregular* but *insensible* when the angles are very small, and they are irregular but large when the angle approaches a right angle, and that with these exceptions the Rule of Proportional Differences is true.

99. The case of the **cosecant.**

The cosecant of an angle is the secant of its complement.

100. The Rule of Proportional Differences is particularly of importance in *practical work.* In Practice the **Logarithms of the Trigonometrical Ratios** are more often used than the Ratios themselves. It is therefore particularly important to consider whether the Rule is true with respect to the Logs of the Trigonometrical Ratios.

101. *To consider the case of the* **L sine.**

$$\log\sin(\theta+\delta)=\log\{\sin\theta+\delta\cos\theta-\tfrac{1}{2}\delta^2\sin\theta-\text{etc.}\}$$
$$=\log\sin\theta\{1+\delta\cot\theta-\tfrac{1}{2}\delta^2-\text{etc.}\}$$
$$=\log\sin\theta+\log\{1+\delta(\cot\theta-\tfrac{1}{2}\delta-\text{etc.})\}$$
$$=\log\sin\theta+\mu\delta(\cot\theta-\tfrac{1}{2}\delta-\ldots)-\tfrac{1}{2}\mu\delta^2(\cot\theta-\tfrac{1}{2}\delta-\ldots)^2+\ldots$$
$$=\log\sin\theta+\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\{1+\cot^2\theta\}+\ldots$$
$$=\log\sin\theta+\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta+\ldots.$$

Hence, omitting higher powers of $\delta$ than $\delta^2$, we have

$$L\sin(\theta+\delta)-L\sin\theta=\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta.$$

If the Tables are calculated for every ten seconds, $\delta$ is not greater than ·00005, and therefore, unless $\cot\theta$ is small or $\operatorname{cosec}^2\theta$ large, we have $L\sin(\theta+\delta) - L\sin\theta = \mu\delta\cot\theta$ as far as seven places of decimals, which proves the Rule to be *generally* true.

102. When $\theta$ is small, $\operatorname{cosec}\theta$ is large. Suppose that the Tables give the $L$ sin of every 10″. Then $\delta$ is not greater than the circular measure of 10″, which is ·0000484... and $\mu$ is not greater than $\frac{1}{2}$. Hence $\frac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta$ is not greater than $\dfrac{6\operatorname{cosec}^2\theta}{10^{10}}$. In order that this may not affect the *seventh* decimal place $6\operatorname{cosec}^2\theta$ must not be greater than $10^3$, that is $\theta$ must not be less than about $5^0$.

Also when $\theta$ is small, $\cot\theta$ is large. Hence when the angles are small, the differences of consecutive $L$ sines are irregular and they are *not* insensible; so that the rule of Proportional Parts does not apply to the $L$ sine when the angle is less than $5^0$.

103. When $\theta$ is nearly a right angle $\cot\theta$ is small and $\operatorname{cosec}\theta$ approaches unity.

Hence when the angles are nearly right angles, the differences of consecutive $L$ sines are irregular but they are at the same time insensible.

104. *The case of the Table of* **L cosines.**

Similar conclusions concerning $\log\cos\theta$ may be inferred from the formula

$$\log\cos(\theta+\delta) = \log\cos\theta + \mu\delta\tan\theta - \tfrac{1}{2}\delta^2\sec^2\theta + \ldots.$$

The differences in this case will be irregular and large when $\theta$ is nearly a right angle, and irregular and insensible when $\theta$ is nearly zero. This is also clear because the sine of an angle is the cosine of its complement.

105. We find then that the Rule of proportional differences cannot be applied to interpolate between the $L$ sines of angles which differ by 10″, when the angle is less than 5°. Three methods have been proposed to replace the Rule.

I. The simplest plan is to have Tables giving the $L$ sines for each second, for the first few degrees of the quadrant.

II. In the following method we require a Table of the same size as that in method I., but it is a Table in which the differences are *insensible.* Accordingly we can with this table calculate the log sine of an angle which lies between two consecutive seconds. The method is as follows:

Let $\theta$ be the circular measure of $n$ seconds. Then when $\theta$ is small $\theta = n \,.\, \sin 1''$ very nearly. Hence

$$\log \frac{\sin\theta}{\theta} = \log \frac{\sin n''}{n \sin 1''} = \log \sin n'' - \log n - \log \sin 1'';$$

$$\therefore L \sin n'' = \log \frac{\sin\theta}{\theta} + L \sin 1'' + \log n$$

$$= \log n + \left( \log \frac{\sin\theta}{\theta} + L \sin 1'' \right).$$

Hence, if a table is constructed giving the values of

$$\left( \log \frac{\sin\theta}{\theta} + L \sin 1'' \right)$$

for every second, for the first few degrees of the quadrant, we can, when the angle is known, find the value of

$$\log \frac{\sin\theta}{\theta} + L \sin 1''$$

from this table, while the value of $\log n$ can be found from the ordinary Table of the logs of numbers. And hence $L \sin n''$ can be found.

Moreover $\frac{\sin\theta}{\theta} = 1 - \frac{1}{6}\theta^2 +$ higher powers of $\theta$, and $\theta$ is by hypothesis the circular measure of an angle less than $5^0$, so that $\frac{\sin\theta}{\theta} = 1 - \frac{1}{6}\theta^2$ approximately. Hence the differences in $\log\frac{\sin\theta}{\theta}$ when $\theta$ is small, will be insensible (*i. e.* will not affect the seventh place of decimals in the result).

[For, $\log\frac{\sin(\theta+\delta)}{\theta+\delta} - \log\frac{\sin\theta}{\theta} = \log\{1 - \frac{1}{6}(\theta+\delta)^2\} - \log(1 - \frac{1}{6}\theta^2)$ the largest term in which, is $-\frac{1}{3}\delta\theta$, *i.e.* the product of two small quantities.]

Therefore we shall not introduce any sensible error in a result obtained from the formula

$$L\sin n'' = \log n + \left(\log\frac{\sin\theta}{\theta} + L\sin 1''\right)$$

if we take the nearest value of $\log\frac{\sin\theta}{\theta} + L\sin 1''$ in the Table. The ordinary table gives the value of $\log n$. Hence we can find $L\sin n''$ even when $n$ is not a whole number.

III. **Maskelyne's Method.** [This method is used in the absence of the special Tables required in I. and II.]

When $\theta$ is small, we have

$$\sin\theta = \theta(1 - \tfrac{1}{6}\theta^2);\ \cos\theta = 1 - \tfrac{1}{2}\theta^2;$$

$$\therefore\ \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = (1 - \tfrac{1}{2}\theta^2)^{\frac{1}{3}} \text{ approximately,}$$

$$= (\cos\theta)^{\frac{1}{3}}, \text{ neglecting higher powers of } \theta \text{ than } \theta^3.$$

Hence $\log\sin\theta = \log\theta + \frac{1}{3}\log\cos\theta.$

Now, when $\theta$ is small the differences of $\log\cos\theta$ are insensible (Art. 104), and if $\theta$ be given we can therefore find $\log\sin\theta$ at once.

If we are given $\log \sin \theta$, we must first find from the table the nearest value of $\theta$, and thence find a value of $\log \cos \theta$ which will not *sensibly* differ from the exact value, and then we get

$$\log \theta = \log \sin \theta - \tfrac{1}{3} \log \cos \theta,$$

and we thus get a second approximation for the value of $\theta$.

With the table of $L$ tangents we proceed thus:

$$\begin{aligned} \log \tan \theta &= \log \sin \theta - \log \cos \theta \\ &= \log \theta - \tfrac{2}{3} \log \cos \theta \text{ approximately from above,} \end{aligned}$$

and this result may be used in a similar manner.

EXAMPLE 1. *Find* L *sin* 1°30′27″·2.

Let $x$ seconds $= \theta$ radians. Then $\pi x = 180 \times 60 \times 60\,\theta$;

$$\therefore \log \theta = \log x + \bar{6}\cdot 6855749.$$

Here $1°30'27''\cdot 2 = 5427\cdot 2$ seconds; $\therefore x = 5427\cdot 2$.

$$\therefore 10 + \log \theta = \log 5427\cdot 2 + 4\cdot 6855749;$$

$$\begin{aligned} \therefore L \sin \theta &= 3\cdot 7345758 + 4\cdot 6855749 - \tfrac{1}{3}(\cdot 0001504) \\ &= 8\cdot 4201006. \end{aligned}$$

EXAMPLE 2. *Find* $\theta$ *when* L *sin* $\theta = 8\cdot 1021832$.

From the Tables by the ordinary Rule we find $\theta = 43'30''$.

Hence if $x$ be the number of seconds in $\theta$

$$10 + \log \theta = \log x + 4\cdot 6855749 = L \sin \theta + \tfrac{1}{3}(L \sec \theta - 10);$$

$$\begin{aligned} \therefore \log x &= 8\cdot 102832 + \bar{5}\cdot 3144251 + \tfrac{1}{3}(\cdot 0000348) \\ &= 3\cdot 4166257 = \log 2609\cdot 88; \end{aligned}$$

$$\therefore \theta = 43'29''\cdot 88.$$

The student should notice the equation

$$10 + \log \theta = \log x + 4\cdot 6855749.$$

## EXAMPLES. XXXII.

(1) Find the following Tabular Logs.

(i) $L \sin 1^0 44' 36''{\cdot}8$. (ii) $L \sin 39' 8''{\cdot}4$. (iii) $L \tan 1^0 44' 36''{\cdot}8$.

(2) Find the angle $\theta$ from the following equations:

(i) $L \sin \theta = 8{\cdot}4832462$. (ii) $L \sin \theta = 8{\cdot}2089620$.

(iii) $L \tan \theta = 8{\cdot}4834473$.

(3) Prove that, if $n$ be the number of seconds in an angle $\theta$

$$L \tan \theta = \log n + 4{\cdot}6855749 + \tfrac{2}{3}(L \sec \theta - 10).$$

106. In practical work it is always advisable to avoid as much as possible that part of a Table in which the differences are insensible. For example, a slight error in the calculation of the sine of an angle nearly $90^0$ would entail a large error in the derived magnitude of the angle. This point is of such great practical importance that we have treated it at some length in the next chapter.

107. The preceding articles afford examples of an important general principle which is of great use in higher mathematics.

If a continuous function of a variable $x$ increases as $x$ approaches a certain value $a$, and begins to diminish directly $x$ has passed the value $a$, then the ratio of the differences of the function to the corresponding small differences in the variable $x$ will diminish and approach to zero as a limit when $x$ approaches $a$.

Thus, $\sin \theta$ is a continuous function of $\theta$ which increases as $\theta$ approaches $\frac{1}{2}\pi$, and it begins to diminish directly $\theta$ has passed through the value $\frac{1}{2}\pi$; hence, as is proved in Art. 92 the ratio of $\sin(\theta + \delta) - \sin \theta$ to $\delta$ tends to become *insensible* as $\theta$ approaches $\frac{1}{2}\pi$.

108. To sum up the Results of this Chapter.

The Rule of Proportional Differences may be used without sensible error in the following cases:

I. For a Table of Common Logarithms giving the logs of all numbers from 10000 to 100000.

II. For a Table of Trigonometrical Ratios calculated for intervals of one minute from 0° to 90°.

Except in the case of

the tangent and secant of angles greater than 45°,

the cotangent and cosecant of angles less than 45°.

III. For a Table of the Tabular Logarithms of Trigonometrical Ratios calculated for intervals of 10″ from 0° to 90°.

Except in the case of

the $L$ sines and $L$ cosecs of angles less than 5°,

the $L$ cosine and $L$ secants of angles greater than 85°,

the $L$ tans and $L$ cotans of angles less than 5° and greater than 85°.

109. The results of this Chapter may also be obtained without actual reference to the *expansions* of $\sin\theta$ and $\cos\theta$ in terms of $\theta$, by the aid of the fact that the difference between $\sin\delta$ and $\delta$ is less than $\frac{1}{6}\delta^3$; so that when $\delta$ is less than 1 degree, $\sin\delta$ and $\delta$ differ by less than ·0000008.

The method of procedure is suggested in the following examples.

## EXAMPLES. XXXIII.

[In these examples $\delta$ is the circular measure of any angle less than 1′.]

(1) Prove that $\sin(\theta+\delta)-\sin\theta=\sin\delta\cos\theta\,(1-\tan\theta\tan\frac{1}{2}\delta)$:

Hence prove that as far as seven places of decimals

$$\sin(\theta+\delta)-\sin\theta=\delta\cos\theta.$$

(2) Prove that $\cos(\theta-\delta)-\cos\theta=\sin\delta\sin\theta\,(1-\cot\theta\tan\frac{1}{2}\delta)$:

Hence prove that as far as seven places of decimals

$$\cos(\theta-\delta)-\cos\theta=\delta\sin\theta.$$

(3) Prove that

$$\tan(\theta+\delta)-\tan\theta=\tan\delta\sec^2\theta\left(\frac{1}{1-\tan\theta\tan\delta}\right):$$

Hence obtain the results of Arts. 95, 96.

(4) Prove that $\cot(\theta-\delta)-\cot\theta=h\operatorname{cosec}^2\theta$ approximately.

(5) Prove that

$$\sec(\theta+\delta)-\sec\theta=\frac{\tan\delta\sin\theta\,(1+\tan\frac{1}{2}\delta\cot\theta)}{\cos^2\theta\,(1-\tan\theta\tan\delta)}:$$

hence prove, except when $\theta$ is small or nearly equal to $\frac{1}{2}\pi$, that

$$\sec(\theta+\delta)-\sec\theta=\delta\sin\theta\sec^2\theta.$$

(6) Prove as in example (5) that

$$\operatorname{cosec}(\theta-\delta)-\operatorname{cosec}\theta=\delta\cos\theta\operatorname{cosec}^2\theta.$$

## CHAPTER IX.

### On Errors in Practical Work.

110. We have already [E. 217, 227] called the student's attention to the approximate nature of all observed measurements.

*Example.* Let the student take any well-defined length, say of 6 or 7 inches, and attempt to ascertain its measure, say to the hundredth part of an inch; and let him repeat the process at another time with different instruments. He will find that unless he makes his measurements with the utmost care, and unless his instruments are very accurately constructed, his two results will in all probability be different.

Such an observation as the above even when made with the greatest care can only be taken as *correct to three significant figures.* If the measurement has to be made correct to a thousandth part of an inch or to any higher degree of accuracy, the student will easily understand that it will be necessary to employ specially constructed instruments. The ordinary diagonal scale or vernier cannot be read with accuracy to the thousandth part of an inch.

111. The student must carefully distinguish between mistakes and errors. By taking sufficient trouble a *calculation* can always be made to attain any required degree of accuracy; so that in what follows we are not concerned with mistakes or inaccuracies in calculation at all.

112. An **error** may be defined as follows.

Suppose an observation made and the result known to be accurate as far as a certain number of significant figures, according to the degree of approximation thought necessary or possible, under the circumstances.

The measure taken may possibly give the magnitude of the quantity with *absolute* accuracy, we cannot say whether it does or not. What we do know is that the difference between the actual magnitude of the quantity and the assumed magnitude is *less than a certain quantity*. This quantity is the *possible error;* and it should be so small that it is either considered of no importance, or is beyond the limit of observation in the circumstances of the case.

113. It is clearly not necessary to carry our calculations to any higher degree of approximation than that represented by the assumed measure.

114. In the practical application of Trigonometry to Engineering and Land Surveying we are concerned with two different kinds of measurements. (1) The measurement of **lines**. (2) The measurement of **angles**. The measurement of a line of any length with anything like the accuracy of five or six significant figures is a very difficult and tedious operation, and is but rarely performed. We know that by the methods of Trigonometry the known length of one line may be made the basis of the calculation of the lengths of all other lines in the survey of a country.

115. The **importance** of an **error in linear measurement** is generally measured by the ratio of the error to the estimated length of the distance under consideration.

*Example.* The problem of calculating the distance of the Sun's centre from that of the earth is beset with such great practical difficulties, that astronomers are only able to say that it is *about* 92 millions of miles. If we knew the distance to *within* a hundred thousand miles, that is, to within about a thousandth part of the distance we should consider the distance to be known with wonderful accuracy. In a distance of this magnitude an error of a few thousands of miles is of *no importance*.

The **importance** of an **error in angular measurement** depends in general simply on the magnitude of the error.

116. If the measure of any **length** is known accurately to seven figures it is **practically exact.** In other words it is known to within the limits of observation.

*Example.* A base line on Salisbury Plain measured with extreme care for the purposes of the Ordnance Survey in England is about 36578 feet in length, and the error is considered to be certainly less than 2 or 3 inches. That is, the error is less than a hundred thousandth part of the whole, and the measurement has been made correct to six significant figures.

The greatest accuracy possible in the measurement of **angles** is attained when the error is known to be not much greater than the tenth part of a second. The tenth part of a second is about the two millionth part of a radian. This degree of accuracy is only attainable under special conditions and with the largest and best instruments.

117. It sometimes happens in the course of a calculation that an error *rises* in importance in consequence of its being multiplied by a very large number.

We may illustrate this by an example.

The height $h$ of a tower is ascertained by measuring a horizontal line $a$ from its base and observing the angle of elevation $\theta$ of the top of the tower from the end of that line.

Then we obtain $h = a \tan \theta$.

Now supposing that we are liable to an error not greater than $\delta$ in the observed magnitude of $\theta$ *we require to know how this will affect the accuracy of the calculated height* $h$.

We know that the error in $\theta$ does not exceed $\delta$. Hence we know that the consequent error in $h$ cannot exeeed $k$

where $h + k = a \tan (\theta + \delta)$.

Hence $k = a \{\tan (\theta + \delta) - \tan \theta\}$

$= a\delta \sec^2 \theta$ neglecting squares and higher powers of $\delta$. [Art. 95.]

Hence the ratio of the error $k$ to the calculated height $h$ is

$$\frac{\delta}{\sin \theta \cos \theta} = \frac{2\delta}{\sin 2\theta}.$$

118. The above result is very instructive.

Suppose the measurements are made with the greatest possible care so that $\delta$ is beyond the limits of observation and may be neglected.

Then we see that in general the importance of the possible 'error' in the calculated height, *i.e.* $2\delta \operatorname{cosec} 2\theta$ is, in general, comparable with $\delta$, and is therefore very small. Also this error is least important when $\operatorname{cosec} 2\theta$ is least, *i.e.* when $\theta$ is $\frac{1}{4}\pi$.

There are two cases however when the error may become of sufficient importance to render the result practically inaccurate. I. when $\theta$ is small, II. when $\theta$ is nearly $\frac{1}{2}\pi$.

In the first case the error itself is not large but it is large *compared with the height to be measured.*

In the second case the error itself is very large, and although the height to be measured is large compared with the base $a$, the importance of the error is also large compared with $\delta$.

It is a difficulty of this latter kind which renders the estimated distance of the sun from the earth so untrustworthy.

119. We have seen that the ratio of the difference in the sine of an angle to the difference in the angle, is small when the angle is nearly $\frac{1}{2}\pi$. That is to say, to a small error in the sine would correspond a large error in the angle. Now, if the sine of an angle has been calculated from observations, and it is found that the value of the sine is nearly unity, we could not without risk of a large error use the value of the angle obtained from the Tables. For, our *observations are known to be liable to errors* (whose magnitude depends on the instruments used, etc.), and therefore the calculated value of the sine under consideration is liable to an error of the same kind. Consequently the calculated value of the angle would be liable to a much *larger error.* And this larger error would possibly affect all results in which the magnitude of the angle was used.

Accordingly in practical work an observer would when possible arrange his measurements so as to avoid such a difficulty—in the working out of a problem—as the necessity for obtaining from the value of its sine, the magnitude of an angle nearly equal to a right angle.

120. The method of Art. 117, which may be applied generally, is of very great importance in practical work; for an observer can often in this way discover beforehand whether any proposed arrangement of his measurements is defective and likely to give unreliable results.

Thus; if the measure of a distance or of any trigonometrical function of an angle be found by means of observed angles and distances, the result is expressed by some formula containing the Trigonometrical functions of the observed angles. If a small error $\delta$ be known or suspected in an observed angle $\theta$, we can find the consequent error in the calculated distance by expanding this formula by the methods of the last chapter in ascending powers of $\delta$. Then, $\delta$ being so small as to be detected with difficulty, $\delta^2$ and higher powers of $\delta$ must be quite beyond the limits of observation. We can in this way estimate the *importance* of a small error in observation.

*Example.* A vertical pole $a$ feet high stands on the top of a cliff, and from a point on the shore the angles of elevation $\alpha$ and $\beta$ of the top and bottom of this pole are observed. The height of the cliff $h$ is given by

$$h = a\frac{\sin\beta\cos\alpha}{\sin(\alpha-\beta)}.$$ [E. LXXII. (9).]

Now suppose an error $\delta$ to have occurred in the observed measurement of the angle $\alpha$, *required the consequent error $h'$ in the calculated height of the cliff.*

We have
$$h+h' = a\frac{\sin\beta\cos(\alpha+\delta)}{\sin(\alpha-\beta+\delta)}$$

$$= \frac{a\sin\beta\{\cos\alpha-\delta\sin\alpha-\text{etc.}\}}{\sin(\alpha-\beta)+\delta\cos(\alpha-\beta)-\text{etc.}}$$

$$= \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}\frac{\{1-\delta\tan\alpha\}}{\{1-\delta\cot(\alpha-\beta)\}} \text{ neglecting } \delta^2 \text{ etc.}$$

$$= \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}\{1-\delta\tan\alpha\}\{1+\delta\cot(\alpha-\beta)\},$$

$$\therefore \qquad h' = \delta \frac{a \sin\beta \cos\alpha}{\sin(\alpha-\beta)} \{\cot(\alpha-\beta) - \tan\alpha\}.$$

Thus the ratio of the error to the estimated height is

$$\delta\{\cot(\alpha-\beta) - \tan\alpha\}.$$

## EXAMPLES. XXXIV.

(1) A triangle is solved from the given parts $A$, $b$, $c$; if there is a small error $\delta$ in the angle $A$ prove that the consequent error in the calculated area of the triangle $B$ is approximately

$$\tfrac{1}{2}\delta bc \cos A.$$

(2) A triangle is solved from the given parts $A$, $b$, $c$; if there is a small error $\delta$ radians in $A$, prove that the consequent error in $B$ is $\qquad -\delta \sin B \cos C \operatorname{cosec} A$ radians.

(3) If the sides of a triangle be measured and a small error $c'$ exist in the measured value of $c$, prove that the consequent error in the diameter of the circumscribing circle is

$$\frac{c' \cos A \cos B}{\sin A \sin B \sin C}.$$

(4) The height and distance of an inaccessible object are found by observing the angles of elevation $\alpha$ and $\beta$ at two points $A$ and $B$ in a horizontal line through the base of the object, the distance between $A$ and $B$ being known; if the same error be made in each in consequence of an imperfect observation of the horizontal, show that the ratio of the error in the calculated height of the object to the calculated distance is

$$\tan(\alpha+\beta) \ : \ 1.$$

(5) The area of a quadrilateral $AOBQ$ right-angled at $A$ and $B$ is to be determined from observations of the angle $AOB$, and the length ($p$ and $q$) of $OA$ and $OB$. Prove that the area is

$$\tfrac{1}{2}\{2ab - (a^2+b^2)\cos\theta\} \operatorname{cosec}\theta,$$

and that if a small error $\delta$ be made in the observation of the angle $AOB$ the consequent error in the area is

$$\tfrac{1}{2}\delta \,.\, AB^2 \,.\, \operatorname{cosec}^2 AOB.$$

(6) If the angles of a triangle, as computed from slightly erroneous measurements of the length of its sides, be $A$, $B$, $C$, prove that approximately, $\alpha\ \beta\ \gamma$ being the errors of the lengths, the consequent errors in the cotangents of the angles are proportional to

$$\beta\cos C+\gamma\cos B-\alpha,\quad \gamma\cos A+\alpha\cos C-\beta,\quad \alpha\cos B+\beta\cos A-\gamma$$

divided respectively by $\sin A$, $\sin B$, $\sin C$.

(7) It is observed that the altitude of the top of a mountain at each of the points $AB$ and $C$ where $ABC$ is a horizontal triangle is $a$. Shew that the height of the mountain is

$$\tfrac{1}{2}a\tan a\operatorname{cosec} A.$$

If there be a small error $n''$ in the altitude at $C$ the true height is very nearly

$$\tfrac{1}{2}\cdot\frac{a\tan a}{\sin A}\left\{1+\frac{\cos C}{\sin A\sin B}\cdot\frac{\sin n''}{\sin 2a}\right\}.$$

(8) If in a triangle $ABC$ the observed lengths of $a$, $b$, $c$ are 5, 4, 6 and there is known to be a small error in the measurement of $c$, determine which angle can be found from the formula

$$\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$$

with the greatest accuracy. [Result. A.]

## CHAPTER X.

### Examples of the Application of Trigonometry to Geometrical Problems.

121. In this chapter we shall use the following notation:

$D$, $E$, $F$ are the feet of the perpendiculars drawn from the angular points $A$, $B$, $C$ of the triangle $ABC$ to the opposite sides.

$AD$, $BE$, $CF$ intersect in a point $P$ which is called the **orthocentre** of the triangle $ABC$.

$DEF$ is called the **pedal triangle** of the triangle $ABC$.

$A'B'C'$ are the middle points of the sides $BC$, $CA$, $AB$.

$AA'$, $BB'$, $CC'$ intersect in a point $G$, which is called the **centre of gravity** of the triangle $ABC$.

$I$, $I_1$, $I_2$, $I_3$ are the centres of the inscribed and escribed circles of the triangle $ABC$; $r$, $r_1$, $r_2$, $r_3$ are their radii.

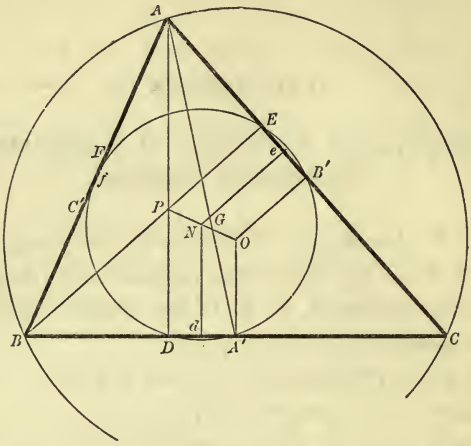
[E. 276, 278.]

$O$ is the centre of the circumscribing circle and $R$ its radius.

The circumscribing circle of the triangle $DEF$ passes through $A'B'C'$ and through the middle points of each of the lines $PA$, $PB$, $PC$. It is called the **nine-points** *circle*. We shall denote its centre by $N$.

[Proofs of the propositions referred to above may be found in the appendix to Todhunter's Euclid.]

EXAMPLE 1. *To prove that* P, N, G *and* O *are in one straight line; and that* PG = 2GO = 4NG, i.e. *that* N *is the point of bisection, and* G *a point of trisection of* PO.




$N$ is the centre of a circle passing through $D$ and $A'$. Therefore $N$ lies in the line bisecting $DA'$ at right angles. This line produced bisects $OP$. Again, the nine-points circle passes through $E$ and $B'$, therefore its centre $N$ lies on the line bisecting $EB'$ at right angles. This line produced also bisects $OP$. Therefore $N$ is the middle point of $OP$.

Again $$AP = \frac{AE}{\cos PAE} = \frac{AE}{\sin C} = \frac{c \cos A}{\sin C};$$

$\therefore$ $$AP = 2R \cos A.$$

But $$OA' = R \cos BOA' = R \cos A;$$

$\therefore$ $$AP = 2OA'.$$

Hence if $AA'$ cut $PO$ in $G$, $AG : GA' = PA : OA' = 2 : 1$;

$\therefore$ $AG = 2GA'$ or $G$ is the centre of Gravity.

Also $$PG : GO = PA : OA' = 2 : 1.$$ Q.E.D.

122. It is often convenient, in attempting the solution of a geometrical problem, to express the lengths of lines involved each in terms of some common unit. When the problem is one concerning a triangle, the Radius of the Circumscribing circle may be employed as the unit. Its convenience is shewn by the symmetry of the following results:—

## EXAMPLES. XXXV.

Prove the following statements:

(1) $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$.

(2) $s=R(\sin A+\sin B+\sin C)=4R\cos\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\cos\frac{1}{2}C$.

(3) $r=4R\sin\frac{1}{2}A\,.\sin\frac{1}{2}B\,.\sin\frac{1}{2}C$.

(4) $r_1=4R\sin\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\cos\frac{1}{2}C$.

(5) $AD=2R\sin B\,.\sin C$. (6) $PD=2R\cos B\,.\cos C$.

(7) $AP=2R\cos A$. (8) $OA'=R\cos A$.

(9) $S=2R^2\sin A\,.\sin B\,.\sin C$.

(10) The radius of the nine-points circle $=\frac{1}{2}R$.

(11) The sides of the triangle $DEF$ are $R\sin 2A$, $R\sin 2B$, $R\sin 2C$.

(12) The area of $DEF=\frac{1}{2}R^2\sin 2A\,.\sin 2B\,.\sin 2C$.

(13) $BD=\frac{1}{2}R(2\cos B\sin C+\sin A)$.

(14) $dN=\frac{1}{2}R\cos(B-C)$.

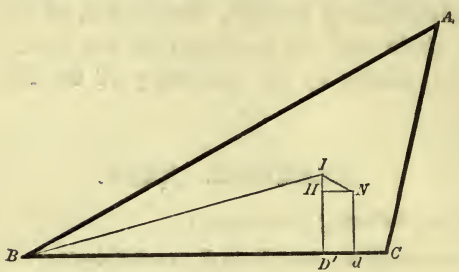
(15) The distances of the centres of the escribed circles from that of the inscribed circles are

$$4R\sin\tfrac{1}{2}A,\quad 4R\sin\tfrac{1}{2}B,\quad 4R\sin\tfrac{1}{2}C.$$

(16) $AE'=R(\sin B+\sin C-\sin A)$, where $E'$ is the point in which the inscribed circle touches $AC$.

EXAMPLE. *To prove that the nine-points circle touches the inscribed circle.*

Draw $ID'$ perpendicular to $BC$ and $NH$ perpendicular to $ID'$.




Then (see Figure on page 117)

$$Nd = \tfrac{1}{2}(PD + OA') = \tfrac{1}{2}(2R\cos B\cos C + R\cos A)$$
$$= \tfrac{1}{2}R\cos(B-C);$$

$$\therefore \quad HI = D'I - dN = r - \tfrac{1}{2}R\cos(B-C) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

Again $Bd = \tfrac{1}{2}(BD + BA') = \tfrac{1}{2}R(2\cos B\sin C + \sin A)$,

and $BD' = S - b = R(\sin A - \sin B + \sin C)$. [*E.* 280.]

$$\therefore \quad HN = \tfrac{1}{2}R(2\cos B\sin C - \sin A + 2\sin B - 2\sin C)$$
$$= R\{\sin B - \sin C - \tfrac{1}{2}\sin(B-C)\} \quad \ldots\ldots\ldots\ldots \text{(II)}.$$

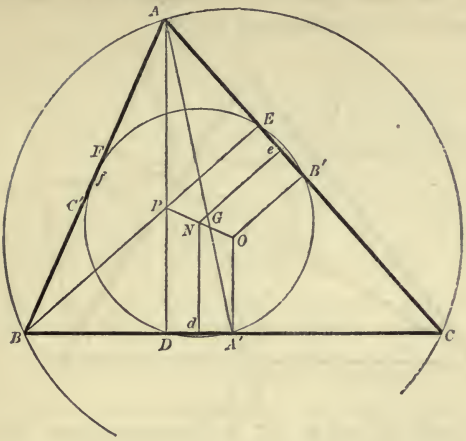
Hence $IN^2 = r^2 + (\tfrac{1}{2}R)^2 - Rr\cos(B-C) + R^2\{(\sin B - \sin C)^2$
$- (\sin B - \sin C)\sin(B-C)\}$.

The last bracket is equal to

$$R^2(\sin B - \sin C)\{2\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(B-C) - 2\sin\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(B-C)\}$$
$$= R^2 4\cos\tfrac{1}{2}(B+C)\sin^2\tfrac{1}{2}(B-C)\{\cos\tfrac{1}{2}(B+C) - \cos\tfrac{1}{2}(B-C)\}$$
$$= -8R^2\sin^2\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}B\,.\,\sin\tfrac{1}{2}C \qquad \text{[xxxv. (3).]}$$
$$= -2Rr\sin^2\tfrac{1}{2}(B-C) = -Rr\{1 - \cos(B-C)\};$$

$$\therefore \quad IN^2 = r^2 + (\tfrac{1}{2}R)^2 - Rr = (r - \tfrac{1}{2}R)^2.$$

That is, the distance between the centres of the circles equals the difference of their radii. Q.E.D.




## EXAMPLES. XXXVI.

Prove the following statements:

(1) The radii of the circles circumscribing $AEF$, $BFD$, $CDE$ are respectively $R\cos A$, $R\cos B$, $R\cos C$.

(2) $AI = 4R\sin\frac{1}{2}B\,.\sin\frac{1}{2}C$.

(3) $AN^2 = \frac{1}{4}R^2(3+2\cos 2A - 2\cos 2B - 2\cos 2C)$.

(4) $OI^2 = R^2 - 2Rr$.

(5) $OI_1^2 = R^2 + 2Rr_1$.

(6) $OP^2 = R^2(1 - 8\cos A\,.\cos B\,.\cos C) = 9R^2 - a^2 - b^2 - c^2$.

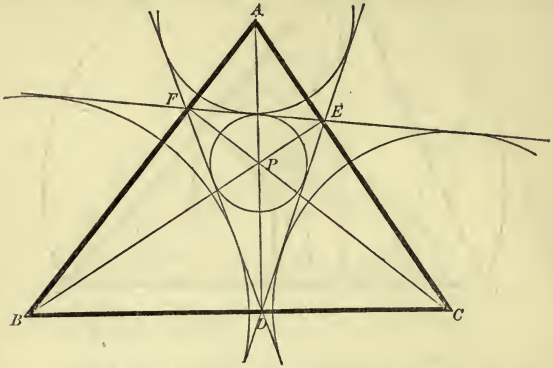
(7) $IP^2 = 4R^2(8\sin^2\frac{1}{2}A\,.\sin^2\frac{1}{2}B\,.\sin^2\frac{1}{2}C - \cos A\,.\cos B\,.\cos C)$.

(8) The area of the triangle

$$IOP = -2R^2\sin\tfrac{1}{2}(B-C)\,.\sin\tfrac{1}{2}(C-A)\,.\sin\tfrac{1}{2}(A-B).$$

(9) $I_1N^2 = (\frac{1}{2}R + r_1)^2$.

123. To prove that the orthocentre is the centre of the circle inscribed in the triangle *DEF*.




The circle of which $PC$ is diameter passes through $E$ and $D$. Therefore the angle $EDC = EPC =$ the complement of $PCE = A$.

Similarly $FDB = FPB = EPC = A$.

Therefore $PDE = 90° - A = PDF$.

Therefore $P$ is the centre of the circle inscribed in $DEF$.

Similarly $A$, $B$, $C$ are the centres of the escribed circles of the triangle $DEF$.

## EXAMPLES. XXXVII.

Prove the following statements:

(1) $ABC$ is the pedal triangle of the triangle $I_1I_2I_3$.

(2) The radius of the circle circumscribing $I_1I_2I_3$ is $2R$.

(3) $DEF$ is the pedal triangle of the triangles $APB$, $BPC$, $CPA$.

(4) The radius of the circumscribing circle of the triangle $APB = R$.

(5) The circle circumscribing $DEF$ touches the circle inscribed in $ABP$.

## MISCELLANEOUS EXAMPLES. XXXVIII.

Prove the following statements :

(1) If a new triangle is formed by joining the centres of the three escribed circles of a triangle $ABC$ the distances of the centres of its escribed circles from the centre of its inscribed circle are

$$8R\sin\tfrac{1}{4}(B+C), \quad 8R\sin\tfrac{1}{4}(C+A), \quad 8R\sin\tfrac{1}{4}(A+B).$$

(2) The areas of the triangles $I_1I_2I_3$, $I_2I_3I$, $I_3II_1$, $II_1I_2$ are to one another inversely as the ratio of $r : r_1 : r_2 : r_3$.

(3) The radii of the escribed circles are the roots of the equation $(x^2+s^2)(x-r)=4Rx^2$.

(4) $PA$, $PB$, $PC$ are the roots of the equation

$$x^3-2(R+r)x^2+(r^2-4R^2+s^2)x-2R\{s^2-(r+2R)^2\}=0.$$

(5) If $p_1$, $p_2$, $p_3$, $-p_4$ are the perpendiculars from $ABC$ and $P$ on the sides of the triangle $DEF$, then $p_1$, $p_2$, $p_3$, $p_4$ are the roots of the equation

$$x^4-2Rx^3+\left(\frac{\Delta^2}{R^2}-2Rr'-r'^2\right)x^2-\frac{\Delta^2}{R^2}r'^2=0,$$

where $r'$ is the radius of the circle inscribed in the triangle $DEF$.

(6) The area of the triangle formed by joining the points of contact of the inscribed circle is

$$2R^2\sin A\,.\sin B\,.\sin C\,.\sin\tfrac{1}{2}A\,.\sin\tfrac{1}{2}B\,.\sin\tfrac{1}{2}C.$$

(7) If the points of contact of each of the four circles touching the three sides of a triangle be joined, and the area of the triangle thus formed from the inscribed circle be subtracted from the sum of the areas of those formed from the escribed circles, the remainder will be double the area of the original triangle.

(8) If $R_1$, $R_2$, $R_3$ are the radii of the circles $BGC$, $CGA$, $AGB$, then

$$\frac{a^2(b^2-c^2)}{R_1^2}+\frac{b^2(c^2-a^2)}{R_2^2}+\frac{c^2(a^2-b^2)}{R_3^2}=0.$$

(9) If $x$ stand for $AI$, $y$ for $BI$, $z$ for $CI$, then
$a^4x^4+b^4y^4+c^4z^4+(a+b+c)^2x^2y^2z^2=2\{b^2c^2y^2z^2+c^2a^2z^2x^2+a^2b^2x^2y^2\}$.

(10) If lines join the points of contact of each escribed circle of a triangle $ABC$ with the produced sides and these lines form a new triangle, then the lines joining the corresponding vertices of the triangle are perpendicular to the sides of the former triangle and are equal to the radii of the escribed circles.

(11) Given the circumscribed and inscribed circles of a triangle, prove that the centres of the escribed circles lie on a fixed circle.

(12) The sum of the reciprocals of the perpendiculars of a triangle is equal to the sum of the reciprocals of the radii of the escribed circles.

(13) If from a point $P$ perpendiculars $PL$, $PM$, $PN$ are drawn to the side of the triangle $ABC$, prove that twice the area of the triangle $LMN=\{R^2-(PO)^2\}\sin A\sin B\sin C$.

(14) The centres of the escribed circles must lie without the circumscribing circle, and cannot be equidistant from it unless the triangle is equilateral.

(15) $r^3I_1 . I_2 . I_3=IA^2 . IB^2 . IC^2$.

(16) The area of the triangle whose angular points are the points of contact of the inscribed circle is to the area of the triangle $ABC$ as $r : 2R$.

(17) If $DEF$ are the points of contact of the inscribed circle with the sides of the triangle $ABC$, then if $AD^2$, $BE^2$, $CF^2$ are in A. P., $a$, $b$, $c$ are in H. P.

(18) From $DEF$ perpendiculars are drawn to the adjacent sides of the triangle $ABC$; prove that the feet of these six perpendiculars lie on a circle whose radius is

$$R(\cos^2A\cos^2B\cos^2C+\sin^2A\sin^2B\sin^2C)^{\frac{1}{2}}.$$

(19) If on one side $BC$ of a triangle $ABC$ a triangle $A'BC$ is described without it such that the angles $BA'C$, $CBA'$, $A'CB$ are equal to $\alpha\beta\gamma$ respectively, then

$$AA'^2 \sin\alpha = \sin\beta \sin\gamma\,(a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma + 4\Delta).$$

(20) If a triangle be cut out in paper and doubled over so that the crease passes through the centre of the circumscribing circle and one of the angles $A$, the area of the doubled portion is $\frac{1}{2}b^2\sin^2 C\cos C\,\mathrm{cosec}\,(2C-B)\sec(C-A)$, $C$ being greater than $B$.

(21) If $IO=IN$, prove that one angle of the triangle $ABC$ is $60^0$.

(22) If two of the angular points and the radius of the circumscribing circle of a triangle are given, the loci of the centre of the nine-point circle and of the ortho-centre are circles.

(23) Prove that a triangle can be constructed whose sides are $a\cos A$, $b\cos B$, $c\cos C$ and that its area is

$$2\Delta\,.\cos A\,.\cos B\,.\cos C.$$

(24) If $R_1$, $R_2$, $R_3$ are the radii of the circumscribed circles of $BIC$, $CIA$, $AIB$, prove that $R_1^2\,.\,R_2^2\,.\,R_3^2 = R^3\,.\,AI\,.\,BI\,.\,CI$.

(25) If the two straight lines which bisect the angles $A$ and $C$ of a triangle $ABC$, meet the circumference of the circumscribing circle in $R$ and $S$, then $RS$ is divided by $CB$, $BA$ into three parts which are in the ratio

$$\sin^2\tfrac{1}{2}A : 2\sin\tfrac{1}{2}A\,.\sin\tfrac{1}{2}B\,.\sin\tfrac{1}{2}C : \sin^2\tfrac{1}{2}C.$$

(26) If a point be taken in an equilateral triangle such that its distances from the angular points are proportional to the sides of a triangle $ABC$, the angles between these distances will be $\frac{1}{3}\pi + A$, $\frac{1}{3}\pi + B$, $\frac{1}{3}\pi + C$.

(27) $$\tan IOI_1 = \pm\frac{2(\sin B - \sin C)}{2\cos A - 1}.$$

## CHAPTER XI.

### On the Use of Subsidiary Angles to facilitate Numerical Calculation.

124. In the *Elementary Trigonometry*, Art. 185, we have shewn how the Tables may be made use of in the solution of Simple Trigonometrical Equations.

It is usual to shew how the Tables may be made use of to facilitate the calculation of the roots of quadratic and cubic equations.

The solution of such equations is however rarely required in practical work, so that the method is not of much practical importance.

125. *To obtain the numerical values of the roots of a* **quadratic equation.**

I. Let the equation be $x^2 - 2px + q = 0$, where $p$ and $q$ are positive.

Solving, we obtain

$$x = p \pm \sqrt{(p^2 - q)}$$

$$= p\left\{1 \pm \sqrt{\left(1 - \frac{q}{p^2}\right)}\right\}.$$

First, let $q$ be less than $p^2$; then we can find from the Tables an angle $\alpha$ such that $\sin^2\alpha = \frac{q}{p^2}$.

Whence we obtain

$$x = p\{1 \pm \cos\alpha\}.$$

Secondly, let $q$ be greater than $p^2$; then we can find from the Tables an angle $\alpha$ such that $\sec^2\alpha = \frac{q}{p^2}$, then

$$x = p\{1 \pm \sqrt{(-1)}\tan\alpha\}.$$

II. Let the equation be $x^2 + 2px + q = 0$. Then the roots of this equation are equal to those in Case I. with the signs changed.

III. Let the equation be $x^2 - 2px - q = 0$.

Solving, we obtain

$$x = p\left\{1 \pm \sqrt{\left(1 + \frac{q}{p^2}\right)}\right\}.$$

We can find from the Tables an angle $\alpha$ such that

$$\tan^2\alpha = \frac{q}{p^2},$$

and then $$x = p\{1 \pm \sec\alpha\}.$$

IV. The roots of $x^2 + 2px - q = 0$ are equal to those of Case III. with the sign changed.

EXAMPLE. *Calculate the value of the roots of the equation*

$$x^2 - 3{\cdot}4651x - 7{\cdot}6842 = 0.$$

Solving, $$x = \frac{3{\cdot}4651}{2}\left\{1 \pm \sqrt{\left(1 + \frac{4 \times 7{\cdot}6842}{(3{\cdot}4651)^2}\right)}\right\}.$$

Hence $$\tan\alpha = \frac{30\cdot7368}{(3\cdot4651)^2};$$

$$\therefore\ L\tan\alpha = \log 30\cdot7368 - 2\log 3\cdot4651 + 10$$

$$= 10\cdot3057240;$$

$$\therefore\ \alpha = 63^0\ 40'\ 55'';$$

$$\therefore\ x = 1\cdot73255\{1 \pm 2\cdot255356\}.$$

126. The student will observe that this method is the same as that of adapting the expression $p\left\{1 \pm \sqrt{\left(1 + \frac{q}{p^2}\right)}\right\}$ to Logarithmic calculation by means of the Trigonometrical Tables.

## EXAMPLES. XXXIX.

Solve the equations :

(i) $x^2 + 3\cdot416x - 8\cdot794 = 0.$

(ii) $x^2 - 7\cdot941x + 2\cdot7001 = 0.$

127. *To obtain the numerical value of the roots of a* **cubic equation.**

Let the equation be

$$x^3 + 3px^2 + 3qx + r = 0.$$

Write $y - p$ for $x$, and the equation becomes

$$y^3 - 3(p^2 - q)y + (2p^3 - 3pq + r) = 0.$$

Therefore any cubic equation can be transformed into another in which the second term is wanting.

Hence we may take as our standard equation

$$x^3 - 3ax + b = 0.$$

To solve this write $ny$ for $x$, thus

$$n^3y - 3any + b = 0,$$

or

$$y^3 - \frac{3ay}{n^2} + \frac{b}{n^3} = 0.$$

But, if $\alpha$ be any angle, we have (E. 167)

$$\cos^3\alpha - \tfrac{3}{4}\cos\alpha - \tfrac{1}{4}\cos 3\alpha = 0.$$

Hence, if we find $\alpha$ such that

$$\cos 3\alpha = -\frac{4b}{n^3}, \text{ while } n = 2\sqrt{a},$$

then $\cos\alpha$ is one of the roots of the equation.

Also since $\cos(2n\pi \pm 3\alpha) = \cos 3\alpha$, the other two values of $y$ are $\cos(\frac{2}{3}\pi + \alpha)$ and $\cos(\frac{2}{3}\pi - \alpha)$.

But $x = ny$. Therefore the required roots are

$$2(a)^{\frac{1}{2}}\cos\alpha, \quad 2(a)^{\frac{1}{2}}\cos(\tfrac{2}{3}\pi + \alpha), \quad 2(a)^{\frac{1}{2}}\cos(\tfrac{2}{3}\pi - \alpha).$$

$3\alpha$ can be found provided $\frac{1}{2}ba^{-\frac{3}{2}}$ is less than unity, *i.e.* provided $b^2$ is less than $4a^3$.

## EXAMPLES. XL.

(1) Solve the following equations:

(i) $x^3 - 3x + 1 = 0.$

(ii) $x^3 - \frac{3}{2}x - \frac{1}{2} = 0.$

(iii) $x^3 - 3x^2 + 3 = 0.$

(iv) $x^3 + 12x^2 + 42x + 44 = 0.$

(v) $x^3 - 3\sqrt{(3)}x^2 - 3x + \sqrt{(3)} = 0.$

(2) Solve the equations:

(i) $x^3 - 439x - 101 = 0$.

(ii) $x^3 - 17651x - 371462 = 0$.

(iii) $1001x^3 - 18472x - 7941 = 0$.

(3) Adapt the following expressions to logarithmic computation:

(i) $a \pm b$. (ii) $a \cos\theta \pm b \sin\theta$.

(iii) $\sin A + \sin B + \sin C - \sin(A+B+C)$.

(iv) $1 + \cos(2\theta - 2m\theta) - \cos(2\theta - 2a) - \cos(2m\theta - 2a)$.

(v) $a \cos A + b \cos B + c \cos C$, where $A$, $B$, $C$ are the angles of a triangle.

(4) If $ke^{i\theta}$ be one of the roots of the equation $x^3 + 3qx + r = 0$ prove that $3q = -k^2(1 + 2\cos 2\theta)$ and $r = 2k^3 \cos\theta$.

128. We shall conclude this chapter with some examples of Elimination.

EXAMPLE I. *Eliminate $\theta$ from the equations*

$$a \cos\theta + b \sin\theta = c,$$

$$a' \cos\theta + b' \sin\theta = c'.$$

Solving these equations, we obtain

$$\cos\theta = \frac{b'c - bc'}{ab' - a'b}, \quad \sin\theta = \frac{c'a - ca'}{ab' - a'b}.$$

But $$\cos^2\theta + \sin^2\theta = 1.$$

Therefore $$(b'c - bc')^2 + (c'a - ca')^2 = (ab' - a'b)^2.$$

This is the required result of elimination.

EXAMPLE II. *Prove that the result of eliminating $\theta$ from the equations*

$$x\cos\theta - y\sin\theta = 2a\cos 2\theta,$$

$$x\sin\theta + y\cos\theta = 2a\sin 2\theta,$$

is $$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

Solving these equations for $x$ and $y$, we obtain

$$x = a\cos 2\theta\cos\theta + 2a\sin 2\theta\sin\theta$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta,$$

$$y = 2a\sin 2\theta\cos\theta - a\cos 2\theta\sin\theta$$

$$= 3a\cos^3\theta\sin\theta + a\sin^3\theta;$$

$$\therefore\quad x+y = a(\cos\theta+\sin\theta)^3,$$

$$x-y = a(\cos\theta-\sin\theta)^3;$$

$$\therefore\quad (x+y)^{\frac{2}{3}} = a^{\frac{2}{3}}\{1+2\cos\theta\sin\theta\},$$

$$(x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}\{1-2\cos\theta\sin\theta\},$$

and the result follows immediately.

## EXAMPLES. XLI.

(1) Given that $$\frac{x^2}{a^2}\cos\theta = \frac{y^2}{a^2}\cos\theta + \frac{z^2}{b^2}\cos\phi,$$

and $$\frac{x}{\sin(\theta+\phi)} = \frac{y}{\sin(\theta-\phi)} = \frac{z}{\sin 2\theta},$$

prove that $$\frac{\sin\theta}{\sin\phi} = \frac{b^2}{a^2}.$$

(2) Eliminate $\theta$ from the equations

$$x = 2a\cos\theta\cos 2\theta - a\cos\theta,$$

$$y = 2b\cos\theta\sin 2\theta - b\sin\theta.$$

(3) Eliminate $a$ and $\beta$ from the equations

$$x=(a\sin^2 a+b\cos^2 a)\cos^2\beta+c\sin^2\beta,$$
$$y=a\cos^2 a+b\sin^2 a,\quad z=(b-a)\sin a\cos a\cos\beta.$$

(4) Eliminate $\theta$ from the equations

$$x+a=a(2\cos\theta-\cos 2\theta),$$
$$y=a(2\sin\theta-\sin 2\theta).$$

(5) Eliminate $\theta$ from the equations

$$b\cos^2\theta+a\sin^2\theta=0,\quad b\cos^2(\theta+\phi)+a\sin^2(\theta+\phi)=0.$$

(6) Eliminate $\theta$ from the equations

$$x=a(\cos\theta+\cos 2\theta),$$
$$y=b(\sin\theta+\sin 2\theta).$$

(7) Eliminate $\theta$ from the equations

$$(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi),$$
$$a\cos 2\phi+b\cos 2\theta=c.$$

(8) Eliminate $\theta$ and $\phi$ from the equations

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2,$$
$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2,$$
$$\theta-\phi=\tfrac{1}{2}\pi.$$

(9) Eliminate $\theta$ from the equations

$$\frac{\cos^3\theta}{\cos(a-3\theta)}=\frac{\sin^3\theta}{\sin(a-3\theta)}=m.$$

(10) Eliminate $a$ from the equations

$$\frac{\sin\theta}{\beta^2-1}=\frac{\cos\theta}{2\beta\sin 2a}=\frac{1}{1+2\beta\cos 2a+\beta^2},$$

shewing that $\beta=\tan\frac{1}{4}(\pi+2\theta)$.

## MISCELLANEOUS EXAMPLES. XLII.

(1) If
$$a\cos(\theta+\phi)+b\cos(\theta-\phi)+c=0,$$
$$a\cos(\phi+\psi)+b\cos(\phi-\psi)+c=0,$$
$$a\cos(\psi+\theta)+b\cos(\psi-\theta)+c=0,$$
then
$$a^2-b^2+2bc=0.$$

(2) If $\cos(y-z)+\cos(z-x)+\cos(x-y)=-\frac{3}{2}$,
then $\cos^3(x+\theta)+\cos^3(y+\theta)$
$$+\cos^3(z+\theta)-3\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)$$
vanishes whatever be the value of $\theta$.

(3) Prove that the equations
$$\left(x+\frac{1}{x}\right)\sin\alpha=\frac{y}{z}+\frac{z}{y}+\cos^2\alpha,$$
$$\left(y+\frac{1}{y}\right)\sin\alpha=\frac{z}{x}+\frac{x}{z}+\cos^2\alpha,$$
$$\left(z+\frac{1}{z}\right)\sin\alpha=\frac{x}{y}+\frac{y}{x}+\cos^2\alpha,$$
are not independent; and that they are equivalent to
$$x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{x}=-\sin\alpha.$$

(4) If $\alpha$ and $\beta$ are the two values of $\sin\theta$ satisfying the equation
$$a\cos 2\ +b\sin 2\theta=c$$
prove that
$$\alpha^2+\beta^2=\frac{a^2-ac+b^2}{a^2+b^2}.$$

(5) If $\sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta$ and
$$a\sin(\theta+\phi)+b\sin(\theta-\phi)=c,$$
then either $a\sin(2\alpha\pm2\beta)=-c$ or $a\sin 2\alpha\pm b\sin 2\beta=c$.

(6) Eliminate $\theta$ from the equations

$$4(\cos a\cos\theta+\cos\phi)(\cos a\sin\theta+\sin\phi)$$
$$=4(\cos a\cos\theta+\cos\psi)(\cos a\sin\theta+\sin\psi)$$
$$=(\cos\phi-\cos\psi)(\sin\phi-\sin\psi),$$

proving that $\cos(\phi-\psi)=1$ or $\cos 2a$.

(7) If $x=2\cos(\beta-\gamma)+\cos(\theta+a)+\cos(\theta-a)$
$=2\cos(\gamma-a)+\cos(\theta+\beta)+\cos(\theta-\beta)$
$=-2\cos(a-\beta)-\cos(\theta+\gamma)-\cos(\theta-\gamma)$,

prove that $x=\sin^2\theta$, provided that the difference between any two of the angles $a$, $\beta$, $\gamma$ neither vanishes nor = a multiple of $2\pi$.

(8) If $\dfrac{\sin(a+\theta)}{\sin(a+\phi)}=\dfrac{\sin(\beta+\theta)}{\sin(\beta+\phi)}$ prove that either $a$ and $\beta$ or $\theta$ and $\phi$ differ by a multiple of $\pi$.

(9) If $\dfrac{\sin(a+\theta)}{\sin(a+\phi)}+\dfrac{\sin(\beta+\theta)}{\sin(\beta+\phi)}=\dfrac{\cos(a+\theta)}{\cos(a+\phi)}+\dfrac{\cos(\beta+\theta)}{\cos(\beta+\phi)}=2$,

prove that either $a$ and $\beta$ differ by an odd multiple of $\frac{1}{2}\pi$, or $\theta$ and $\phi$ differ by an even multiple of $\pi$.

(10) If $(A+B+C)=\pi$ and if $\cos 2A=\dfrac{(d-a)(b-c)}{(d+a)(b+c)}$,

$\cos 2B=\dfrac{(d-b)(c-a)}{(d+b)(c+a)}$, $\cos 2C=\dfrac{(d-c)(a-b)}{(d+c)(a+b)}$,

then $\tan A+\tan B+\tan C=\pm 1$.

(11) If $\cos a=\cos\beta\cos\phi=\cos\beta'\cos\phi'$,

and $\sin a=2\sin\dfrac{\phi}{2}\sin\dfrac{\phi'}{2}$,

then $\tan\dfrac{a}{2}=\tan\dfrac{\beta}{2}\cdot\tan\dfrac{\beta'}{2}$.

(12) If $\tan\phi=\dfrac{\sin a\sin\theta}{\cos\theta-\cos a}$, then $\tan\theta=\dfrac{\sin a\sin\phi}{\cos\phi\pm\cos a}$.

(13) If $\beta$ and $\gamma$ be two values of $\theta$ which satisfy

$$\frac{1}{a}\cos\theta+\frac{1}{b}\sin\theta=\frac{1}{c},$$

then $$a\cos\frac{\beta+\gamma}{2}=b\sin\frac{\beta+\gamma}{2}=c\cos\frac{\beta-\gamma}{2}.$$

(14) Given $$a^2\cos\alpha\cos\beta+a(\sin\alpha+\sin\beta)+1=0,$$

$$a^2\cos\alpha\cos\gamma+a(\sin\alpha+\sin\gamma)+1=0,$$

prove that $$a^2\cos\beta\cos\gamma+a(\sin\beta+\sin\gamma)+1=0,$$

and that $$\cos\alpha+\cos\beta+\cos\gamma=\cos(\alpha+\beta+\gamma),$$

$\beta$ and $\gamma$ being unequal and less than $\pi$.

(15) If $\theta_1$ and $\theta_2$ are two values of $\theta$ which satisfy

$$1+\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}=0,$$

shew that $\theta_1$ and $\theta_2$ if substituted for $\theta$ and $\phi$ in the equation will satisfy it.

(16) Solve the equations

$$\cos(\theta+\alpha)=\sin\phi\sin\beta,$$

$$\cos(\phi+\beta)=\sin\theta\sin\alpha,$$

and shew that if $\phi_1$ and $\phi_2$ be the two values of $\phi$,

$$\tan(\phi_1+\phi_2)=\frac{\sin 2\beta}{\sin^2\beta-\cos^2\beta\sec^2\alpha}.$$

(17) If $$\cos(\alpha+\theta)+m\cos\theta=n,$$

$n^2$ cannot be greater than $1+2m\cos\alpha+m^2$.

(18) Eliminate $\theta$ and $\phi$ from the equations

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1,$$

$$\theta-\phi=2\alpha,$$

proving that $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1-\cos 2\alpha}.$$

## GENERAL MISCELLANEOUS EXAMPLES. XLIII.

N.B.—For convenience in printing, some writers use $n!$ to denote $1.2.3...n$.

(1) A person walks from one end $A$ of a wall a certain distance $a$ towards the West, and observes that the other end $B$ then bears E.S.E. He afterwards walks from the end $B$ a distance $(\sqrt{2}+1)a$ towards the South, and finds that the end $A$ bears N.W. Shew that the wall makes an angle $\cot^{-1}2$ with the East.

(2) A man on the top of a hill observes the angles of depression $\alpha, \beta, \gamma$ of three consecutive milestones on a straight horizontal road running directly towards him; prove that the height of the hill is

$$1760\left(\frac{\sqrt{2}}{\cot^2\alpha - 2\cot^2\beta + \cot^2\gamma}\right)^{\frac{1}{2}} \text{ yards.}$$

(3) $\sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta)$

$=4\sin\alpha\sin\beta\sin\gamma\cos(\alpha+\beta+\gamma)+4\cos\alpha\cos\beta\cos\gamma\sin(\alpha+\beta+\gamma).$

(4) If $2\alpha+2\beta+2\gamma=n\pi$,

then $\sin 2(\beta+\gamma)+\sin 2(\gamma+\alpha)+\sin 2(\alpha+\beta)$

$=2(-1)^{\frac{n-1}{2}}\{1-(-1)^n\}\cos\alpha\cos\beta\cos\gamma$

$+2(-1)^{\frac{n}{2}}\{1+(-1)^n\}\sin\alpha\sin\beta\sin\gamma.$

(5) $$\frac{\log_a \log_a N}{\sqrt{\log_a b}} - \frac{\log_b \log_b N}{\sqrt{\log_b a}} = \frac{\log_a \log_a b}{\sqrt{\log_a b}} = -\frac{\log_b \log_b a}{\sqrt{\log_b a}}.$$

(6) $$\frac{1}{\lfloor 2} + \frac{4}{\lfloor 3} + \frac{9}{\lfloor 4} + \frac{16}{\lfloor 5} + \ldots = e - 1.$$

(7) If $A+B+C=\pi$, and

$$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta),$$

then will $$\cot\theta = \cot A + \cot B + \cot C.$$

(8) Eliminate $\theta$ from the equations

$$(a+b)(x+y) = \cos\theta(1+2\sin^2\theta),$$
$$(a-b)(x-y) = \sin\theta(1+2\cos^2\theta).$$

(9) If $\cos(\theta-\phi)$ is a mean proportional between $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$, then

$$\operatorname{cosec} 2\theta + \sec 2\theta = \operatorname{cosec} 2\phi + \sec 2\phi.$$

(10) If $A+B+C=90^0$, then $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C - 2$

$$= \cot B \tan C + \cot C \tan B + \cot C \tan A + \cot A \tan C + \cot A \tan B + \cot B \tan A.$$

(11) If $\tan(B+C-A)\tan(C+A-B)\tan(A+B-C)=1$,

then $$\sin 4A \sin 4B \sin 4C = 4\cos 2A \cos 2B \cos 2C.$$

(12) If $4(\alpha+\beta+\gamma)=\pi$, prove that

$$\cos(6\beta+4\gamma-8\alpha) + \cos(6\gamma+4\alpha-8\beta) + \cos(6\alpha+4\beta-8\gamma)$$
$$= 4\cos(5\alpha-2\beta-\gamma)\cos(5\beta-2\gamma-\alpha)\cos(5\gamma-2\alpha-\beta).$$

(13) If $$1-x^2-y^2-z^2 = 2xyz,$$

prove by trigonometry, that

$$x\sqrt{(1-x^2)} + y\sqrt{(1-y^2)} + z\sqrt{(1-z^2)} = 2\sqrt{\{(1-x^2)(1-y^2)(1-z^2\}}.$$

(14) The formulae

$$(2n+\tfrac{1}{4})\pi \pm \alpha, \quad (n-\tfrac{1}{4})\pi + (-1)^n(\tfrac{1}{2}\pi - \alpha)$$

represent the same series of angles.

(15) If $A+B+C=(2m+1)\pi$ or $2m\pi+\frac{1}{2}\pi$, then

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)=2\sin A\,.\,\sin B\,.\,\sin C+2\cos A\cos B\,.\,\cos C+1;$$

and if $A+B+C=2m\pi$ or $2m\pi-\frac{1}{2}\pi$, then

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)=2\sin A\sin B\sin C+2\cos A\cos B\cos C-1.$$

(16) If $\sin(2n+1)A\sin(B-C)+\sin(2n+1)B\sin(C-A)+\sin(2n+1)C\sin(A-B)=0$, where $n$ is an integer, then

$$\sin(n-1)A\sin(n+1)(B-C)+\sin(n-1)B\sin(n+1)(C-A)+\sin(n-1)C\sin(n+1)(A-B)=0.$$

(17) If $\sin 2\theta+\sin 2\phi=\sin 2a$, prove that the three expressions

$$\cos\theta\cos(a+\theta)\sin\theta\sin(a-\theta),$$
$$\cos\phi\cos(a+\phi)\sin\phi\sin(a-\phi),$$
$$\cos\theta\cos\phi\sin\theta\sin\phi,$$

are equal to one another.

(18) If $a+\beta+\gamma=m\pi$, then

$$\frac{\cos\beta\cos(2\gamma-\beta)}{1+8\cos\beta\cos\gamma\cos(\beta+\gamma)}+\frac{\cos\gamma\cos(2a-\gamma)}{1+8\cos\gamma\cos a\cos(\gamma+a)}+\frac{\cos a\cos(2\beta-a)}{1+8\cos a\cos\beta\cos(a+\beta)}=0.$$

(19) If $\dfrac{\cos(a+\beta+\theta)}{\sin(a+\beta)\cos^2\gamma}=\dfrac{\cos(\gamma+a+\theta)}{\sin(\gamma+a)\cos^2\beta}$ and $\beta$ and $\gamma$ are unequal, then each of these fractions is equal to $\dfrac{\cos(\gamma+\beta+\theta)}{\sin(\gamma+\beta)\cos^2 a}$, and $\cos\theta=\dfrac{\sin(\beta+\gamma)\sin(\gamma+a)\sin(a+\beta)}{\cos(\beta+\gamma)\cos(\gamma+a)\cos(a+\beta)+\sin^2(a+\beta+\gamma)}$.

(20) If
$$\sqrt{2}\cos A=\cos B+\cos^3 B,$$
$$\sqrt{2}\sin A=\sin B-\sin^3 B,$$
then
$$\pm\sin(B-A)=\cos 2B=\tfrac{1}{3}.$$

(21) If $4\cos(x-y)\cos(y-z)\cos(z-x)=1$, prove that

$$1+12\cos 2(x-y)\cos 2(y-z)\cos 2(z-x)$$
$$=4\cos 3(x-y)\cos 3(y-z)\cos 3(z-x).$$

(22) If

$$\sin(\beta+\gamma)-k\sin(\alpha+\delta)=\sin(\gamma+\alpha)-k\sin(\beta+\delta)$$
$$=\sin(\alpha+\beta)-k\sin(\gamma+\delta),$$

where $\alpha$, $\beta$, $\gamma$ are unequal and each less than $\pi$, then will $k^2=1$, and each member of the equations $=0$.

(23) If $\dfrac{\cos(\beta+\alpha)+\cos(\alpha+\gamma)}{\cos(\beta-\alpha)+\cos(\alpha-\gamma)}=\dfrac{\cos(\beta+\gamma)+\cos(\alpha+\gamma)}{\cos(\gamma-\beta)+\cos(\alpha-\gamma)}$,

then $$\frac{\sin\beta}{\cos\gamma-\cos\alpha}+\frac{\cos\beta}{\sin\alpha-\sin\gamma}=\frac{1}{\sin(\alpha-\gamma)}.$$

(24) If $x^2\cos\alpha\cos\beta+x(\sin\alpha+\sin\beta)+1=0$,

and $x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0$,

prove that $x^2\cos\gamma\cos\alpha+x(\sin\gamma+\sin\alpha)+1=0$.

(25) If
$$x+y\cos\alpha+z\sin\alpha=\cos(\beta-\gamma),$$
$$x+y\cos\beta+z\sin\beta=\cos(\gamma-\alpha),$$
$$x+y\cos\gamma+z\sin\gamma=\cos(\alpha-\beta),$$

prove that $x=4\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)$.

(26) If
$$a\cos\alpha+b\cos\beta+c\cos\gamma=0,$$
$$a\sin\alpha+b\sin\beta+c\sin\gamma=0,$$
$$a\sec\alpha+b\sec\beta+c\sec\gamma=0,$$

then $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2=0$.

(27) If
$$a\cos\theta+b\sin\theta+c=0,$$
$$a\cos\phi+b\sin\phi+c=0,$$
$$c\sqrt{2}=\pm a\pm b,$$

prove that either $\theta$ or $\phi$ must be of the form $\frac{1}{2}n\pi+\frac{1}{4}\pi$.

(28) If $m$ is a positive integer, then

$$\cos(m+1)\theta = \cos m\theta \left\{ 2\cos\theta - \frac{1}{2\cos\theta -} \quad \frac{1}{2\cos\theta -} \cdots\cdots \frac{1}{\cos\theta} \right\},$$

where $2\cos\theta$ is repeated $m$ times.

(29) If $2s = x+y+z$, prove

(i) $$\tan(s-x)+\tan(s-y)+\tan(s-z)-\tan s = \frac{4\sin x \sin y \sin z}{1-\cos^2 x-\cos^2 y-\cos^2 z+2\cos x\cos y\cos z},$$

(ii) $$\tan^{-1}(s-x)+\tan^{-1}(s-y)+\tan^{-1}(s-z)-\tan^{-1}s = \tan^{-1}\frac{16xyz}{(x^2+y^2+z^2+4)-4(y^2z^2+z^2x^2+x^2y^2)}.$$

(30) (i) $$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} + \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} = 1,$$

(ii) $$\frac{\sin(\theta-\alpha)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{\sin(\theta-\beta)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} + \frac{\sin(\theta-\gamma)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} = 0.$$

(31) If $a^2$, $b^2$, $c^2$ are in A. P., then $\tan A$, $\tan B$, $\tan C$ are in H. P.

(32) If $\dfrac{\tan A}{x} = \dfrac{\tan B}{y} = \dfrac{\tan C}{z}$, prove that each fraction is equal to

$$\sin A \,.\, \sin B \,.\, \sin C \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{x+y+z} \right).$$

(33) $OI^2 + OI_1^2 + OI_2^2 + OI_3^2 = 12R^2$.

(34) If $\alpha$, $\beta$, $\gamma$ are the radii of the circles $OAB$, $OBC$, $OCA$, then

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = \frac{abc}{R^3}.$$

(35) The sides of a triangle are

$$x^2+xy+y^2,\ 2xy+x^2,\ x^2-y^2;$$

prove that the angles are in A.P., the common difference being

$$2\tan^{-1}\left(\frac{y\sqrt{3}}{2x+y}\right).$$

(36) $4R(r_2r_3+r_3r_1+r_1r_2)=(r_2+r_3)(r_3+r_1)(r_1+r_2)$.

(37) $\Delta=s^2 2^{\frac{3}{2}}\cos\theta\cos\phi\cos\psi$, where

$$\cos 2\theta=\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C,$$
$$\cos 2\phi=\tan\tfrac{1}{2}C+\tan\tfrac{1}{2}A,$$
$$\cos 2\psi=\tan\tfrac{1}{2}A+\tan\tfrac{1}{2}B.$$

(38) $4(AA'^2+BB'^2+CC'^2)=3(a^2+b^2+c^2)$.

$16(BB'^2.CC'^2+CC'^2.AA'^2+AA'^2.BB'^2)=9(b^2c^2+c^2a^2+a^2b^2)$.

$16(AA'^4+BB'^4+CC'^4)=9(a^4+b^4+c^4)$. [Art. 121.]

(39) If $a, b, c, d$ be the lengths of the sides of a quadrilateral such that one circle can be described about it and another inscribed in it, then the radius of the latter circle is

$$\frac{2\sqrt{(abcd)}}{a+b+c+d}.$$

(40) Given that $x=y\cos Z+z\cos Y$,

$$y=z\cos X+x\cos Z,$$

and that $X+Y+Z$ is an odd multiple of $\pi$, then

$$z=x\cos Y+y\cos X,$$

and

$$\cos X=\frac{x^2+y^2-z^2}{2yz}.$$

(41) If $A+B+C=180^0$ and

$$\frac{y\sin C-z\sin B}{x-y\cos C-z\cos B}=\frac{z\sin A-x\sin C}{y-z\cos A-x\cos C},$$

$x, y, z$ being real, then $\dfrac{x}{\sin A}=\dfrac{y}{\sin B}=\dfrac{z}{\sin C}$.

(42) In any triangle

$$R^2(a^4+b^4+c^4-2b^2c^2\cos 2A-2c^2a^2\cos 2B-2a^2b^2\cos 2C)=2a^2b^2c^2.$$

(43) The radii of the escribed circles of a triangle are the roots of the equation

$$Sx^3+s(\Sigma-s^2)x^2+s^2Sx=\Delta^2 s,$$

where $2\Sigma$ is the sum of the squares of the sides.

(44) $12RS=a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$.

(45) If $O$ be a point within a triangle, such that $AO$, $BO$, $CO$ are inversely proportional to the sides $BC$, $CA$, $AB$; $R$, $R_1$, $R_2$, $R_3$ are the radii of the circles described about $ABC$, $BOC$, $COA$, $AOB$; then

$$\left(\frac{2}{R_1}-\frac{1}{R}\right)\tan A=\left(\frac{2}{R_2}-\frac{1}{R}\right)\tan B=\left(\frac{2}{R_3}-\frac{1}{R}\right)\tan C.$$

(46) $ABC$ is a triangle in which the sides $AB$, $AC$ are equal. Circles are described with centres $A$, $B$, $C$ touching each other externally. Prove that the distance between the centres of the circles that can be drawn touching these three circles is

$$4b\,\frac{(1-\cos B)(1-2\cos B)}{4-5\cos B}.$$

(47) Perpendiculars $OD$, $OE$, $OF$ to the sides of a triangle when produced meet the circumscribing circle in $P$, $Q$, $R$; prove that

$$4r(PD+QE+RF)=2bc+2ca+2ab-a^2-b^2-c^2.$$

(48) If $\rho_1$, $\rho_2$, $\rho_3$ are the distances of any point in the plane of an equilateral triangle, whose side is $a$, from the angular points, then

$$\rho_2^2\rho_3^2+\rho_3^2\rho_1^2+\rho_1^2\rho_2^2-\rho_1^4-\rho_2^4-\rho_3^4+a^2(\rho_1^2+\rho_2^2+\rho_3^2)=a^4.$$

(49) From the angular points of a triangle $ABC$ are drawn perpendiculars to the opposite sides and also lines bisecting the angles: if $\theta$ be the angle between the two lines drawn from $A$, and $\phi$, $\psi$ be corresponding angles at $B$, $C$, prove that

$$1+\cos\theta+\cos\phi+\cos\psi=4\cos\tfrac{1}{4}(B-C)\cos\tfrac{1}{4}(C-A)\cos\tfrac{1}{4}(A-B).$$

(50) On the sides of a scalene triangle $ABC$ as bases similar isosceles triangles are described either all internally or all externally, and their vertices joined so as to form a new triangle $A'B'C'$; then if $A'B'C'$ is equilateral the angles at the bases of the isosceles triangles will be each $30^0$, and if it is similar to $ABC$ they are each $\tan^{-1}\frac{4\Delta}{a^2+b^2+c^2}$.

(51) Three circles touching each other externally are all touched by a fourth circle including them all. If $a$, $b$, $c$ are the radii of the internal circles and $\alpha$, $\beta$, $\gamma$ the distance of their centres from that of the external circle, then

$$2\left(\frac{\beta\gamma}{bc}+\frac{\gamma\alpha}{ca}+\frac{\alpha\beta}{ab}\right)=4+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}.$$

(52) Circles are described on the sides $a$, $b$, $c$ of a triangle as diameters; prove that the diameter $D$ of the circle which touches them externally is such that

$$\left(\frac{D}{s-a}-1\right)^{\frac{1}{2}}+\left(\frac{D}{s-b}-1\right)^{\frac{1}{2}}+\left(\frac{D}{s-c}-1\right)^{\frac{1}{2}}=\left(\frac{s}{D-s}\right)^{\frac{1}{2}}.$$

(53) If $O$ be any point and $\rho_1$, $\rho_2$, $\rho_3$ the reciprocals of the radii of the circles circumscribing the three triangles $OBC$, $OCA$, $OAB$, prove that

$$(a\rho_1+b\rho_2+c\rho_3)(-a\rho_1+b\rho_2+c\rho_3)(a\rho_1-b\rho_2+c\rho_3)(a\rho_1+b\rho_2-c\rho_3)$$
$$=a^2\rho_1^2b^2\rho_2^2c^2\rho_3^2.$$

(54) Lines drawn parallel to the sides of a triangle $ABC$ through the centres of the circles escribed to that triangle form a triangle $A'B'C'$; prove that the perimeter of the triangle $A'B'C'$ is

$$4R\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C.$$

(55) If $BE$, $CF$ be the perpendiculars from $B$ and $C$ on to the opposite sides, and if $FE$ and $BC$ produced meet in $Q$, prove that

$$2(QE^2-QF^2)=(BQ^2-CQ^2)(\cos 2B+\cos 2C).$$

(56) If $O$ be any point within a triangle $ABC$ the sides of which are $abc$, and if $R_1$, $R_2$ be the radii of the circles circumscribing the triangles $BOC$, $COA$, prove that

$$\frac{1}{R_1 a a} - \frac{1}{R_2 b \beta} = \frac{1}{a\beta\gamma}\left(\frac{\beta^2+\gamma^2}{a^2} - \frac{\gamma^2+a^2}{b^2}\right),$$

where $$OA=a, \quad OB=\beta, \quad OC=\gamma.$$

(57) A chord is drawn cutting two concentric circles whose radii are as $1 : n$ so that the intercepted portions subtend angles $2\alpha$, $2\beta$ at the centre; prove that the chord is divided at either point of intersection with the inner circle in the ratio

$$n^2 - 2n\cos(\alpha+\beta) + 1 : n^2 - 1.$$

(58) A straight line cuts three concentric circles in $A$, $B$, $C$ and passes at a distance $p$ from their centre. The area of the triangle formed by the tangents at $ABC$ is $\dfrac{BC \cdot CA \cdot AB}{2p}$.

(59) A polygon of $3n$ sides, which are $a$, $b$, $c$ successively, repeated $n$ times, is inscribed in a circle : if the angular points be $A$, $B$, $C$, $D$, $E$, etc., and the radius of the circle is denoted by $r$, prove that

$$AC^2 = \left\{ac + 2br\sin\frac{\pi}{n}\right\}\left\{bc + 2ar\sin\frac{\pi}{n}\right\} \div \left\{ab + 2cr\sin\frac{\pi}{n}\right\}.$$

(60) The tangent at the point of contact of the inscribed circle and the circle of nine points of the triangle $ABC$ cuts the side $AC$ at a distance from $A$

$$\frac{b(a-b)}{a-2b+c}.$$

(61) $\tan x = kx$ has an infinity of roots.

(62) The equation $\theta = \cos\theta$ has one and only one solution, such that the value of $\theta$ is less than $\frac{1}{4}\pi$.

(63) $8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ is less than 1 except when the triangle $ABC$ is equilateral.

(64) If $A+B+C=90^0$, the least value of

$$\tan^2 A+\tan^2 B+\tan^2 C \text{ is } 1.$$

(65) If $A+B+C=90^0$

$$\tan^2 B\tan^2 C+\tan^2 C\tan^2 A+\tan^2 A\tan^2 B$$

is always less than 1; and if one angle approach indefinitely near to two right angles the least value of the expression is $\frac{1}{2}$.

(66) If $A+B+C=180^0$, the least value of

$$\cot^2 A+\cot^2 B+\cot^2 C \text{ is } 1.$$

(67) In any triangle $\cos A+\cos B+\cos C$ is $>1$ and not greater than $\frac{3}{2}$.

(68) $\sin A+\sin B+\sin C$ is never less than

$$\sin 2A+\sin 2B+\sin 2C, \text{ if } A+B+C=180^0.$$

(69) $$\left(2\cos\alpha-\frac{1}{2\cos\alpha-}\,\frac{1}{2\cos\alpha-}\cdots\right)\left(2\cos\beta-\frac{1}{2\cos\beta-}\,\frac{1}{2\cos\beta-}\cdots\right)$$
$$=2\cos(\alpha+\beta)-\frac{1}{2\cos(\alpha+\beta)-}\,\frac{1}{2\cos(\alpha+\beta)-}.$$

(70) Prove by induction that

$$\sin(\alpha+\beta+\gamma+\ldots n \text{ angles})=(S_1C_{n-1})-(S_3C_{n-3})+(S_5C_{n-5})-\text{etc.}$$
$$\cos(\alpha+\beta+\gamma+\ldots n \text{ angles})=(C_n)-(S_2C_{n-2})+(S_4C_{n-4})-\text{etc.}$$

where $(S_rC_{n-r})$ stands for the sum of the products of the sines $\sin\alpha$, $\sin\beta$, $\sin\gamma\ldots$ taken $r$ together, each multiplied by the product of the $(n-r)$ remaining cosines.

Shew that De Moivre's Theorem is equivalent to these two theorems.

(71) Prove that if $n$ is an odd integer the two series

$$1-\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}-\ldots,$$

$$n-\frac{n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}-\ldots,$$

are numerically equal, and if $n$ is an even integer one of the two series is zero.

(72) Prove that $\dfrac{3}{1+x^3}=\dfrac{1}{1+x}+\dfrac{1}{1-\alpha x}+\dfrac{1}{1-\beta x}$,

where $-1$, $\alpha$, $\beta$ are the values of $(-1)^{\frac{1}{3}}$, and deduce by writing $x=\cos 2\theta+i\sin 2\theta$ that

$$3\tan 3\theta=\tan\theta-\cot(\theta+\tfrac{1}{6}\pi)-\cot(\theta-\tfrac{1}{6}\pi).$$

(73) If $\quad \sin\log(a+ib)=\alpha+i\beta,$

then $\quad \log\sin(\alpha'+ib')=\alpha'+i\beta';$

when $\quad \alpha'=\log\sqrt{(a^2+b^2)},\quad b'=\tan^{-1}\dfrac{b}{a},$

$$\alpha'=\log\sqrt{(\alpha^2+\beta^2)},\quad \beta'=\tan^{-1}\frac{\beta}{\alpha}.$$

(74) By writing in the identity

$$\frac{1}{(x-a)(x-b)}=\frac{1}{(a-b)(x-a)}-\frac{1}{(a-b)(x-b)},$$

$\cos 2\theta+\sqrt{(-1)}\sin 2\theta$ for $x$ and similar quantities in terms of $\alpha$ and $\beta$ for $a$ and $b$, prove that

$$\cos(2\theta+\alpha+\beta)\sin(\alpha-\beta)=\cos(2\alpha+\theta+\beta)\sin(\theta-\beta)$$
$$-\cos(2\beta+\theta+\alpha)\sin(\theta).$$

(75) Prove that

$$\cos(x+iy)=\cos x\cosh y-i\sin x\sinh y,$$

$$\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y.$$

(76) If $2x=4\cos a\cosh b$, $2y=4\sin a\sinh b$, then

$$\sec(a+ib)+\sec(a-ib)=\frac{4x}{x^2+y^2},$$

$$\sec(a+ib)-\sec(a-ib)=\frac{4iy}{x^2+y^2}.$$

(77) One of the values of $\sin^{-1}(\cos\theta+i\sin\theta)$ is

$$\cos^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{\sin\theta}+\sqrt{(1+\sin\theta)}\}$$

when $\theta$ is between 0 and $\frac{1}{2}\pi$.

(78) Reduce $\tan^{-1}(\cos\theta+i\sin\theta)$ to the form $A+iB$, and hence prove that $\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta-\ldots=\pm\frac{1}{4}\pi$; the upper or lower sign being taken according as $\theta$ is positive or negative.

(79) If $\omega$, $\omega^2$ are the imaginary cube roots of $-1$,

$$\left\{\cos a+\omega\cos\left(a+\frac{\pi}{3}\right)+\omega^2\cos\left(a+\frac{2\pi}{3}\right)\right\}$$
$$\left\{\cos\beta+\omega\cos\left(\beta+\frac{\pi}{3}\right)+\omega^2\cos\left(\beta+\frac{2\pi}{3}\right)\right\}$$
$$=\tfrac{3}{2}\left\{\cos(a+\beta)+\omega\cos\left(a+\beta+\frac{\pi}{3}\right)+\omega^2\cos\left(a+\beta+\frac{2\pi}{3}\right)\right\},$$

and deduce the value of

$$\left\{\cos a+\omega\cos\left(a+\frac{\pi}{3}\right)+\omega^2\cos\left(a+\frac{2\pi}{3}\right)\right\}^n,$$

when $n$ is a positive integer.

(80) Prove that the real part of $(1+i\tan\theta)^{-i}$ is

$$e^{\theta}\cos(\log\cos\theta).$$

(81) $(a+ib)^{a+i\beta}$ will be wholly real or wholly imaginary according as $\frac{1}{2}\beta\log(a^2+b^2)+a\tan^{-1}\frac{b}{a}$ is an even or odd multiple of $\frac{1}{2}\pi$.

(82) Prove that all solutions of the equation $\sinh x=\sinh a$ are included in the expression $x=n\pi+(-1)^n a$ where $n$ is any integer, positive or negative.

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(83) Prove the following rule for finding the length of a small circular arc; to 256 times the chord of one-fourth the arc add the chord of half the arc; subtract 40 times the chord of half the arc and divide the remainder by 45.

(84) If two sides $a$, $b$ and the included angle $C$ of a triangle are given, and a small error $\delta$ exists in $C$, the corresponding error in $R$ is $\frac{1}{2}\delta a \cot A \cos B \operatorname{cosec} C$.

(85) If the unit of measurement be a right angle, find the limit of $\dfrac{\tan\theta - \sin\theta}{\theta^3}$ as $\theta$ is indef. diminished.

(86) The limit when $n$ is indefinitely increased of

$$\left(\cos\frac{a}{n} + \sin\frac{3a}{n}\right)^n \text{ is } e^{3a}.$$

(87) If $\dfrac{\sin\theta}{\theta} = \dfrac{863}{864}$, then $\theta = 5^0$ nearly.

(88) If $\theta = \frac{1}{2}\pi$ nearly and $n$ is $> 1$, prove that

$$(\sin\theta)^{\frac{1}{n}} = \frac{n-1+(n+1)\sin\theta}{n+1+(n-1)\sin\theta} \text{ nearly.}$$

(89) If $\cos^{-1}\dfrac{a}{a+b+x} - \cos^{-1}\dfrac{a}{a+x} = \gamma$, and $b$ and $x$ are both small compared with $a$, then

$$x = \frac{a\sin^2\gamma}{8}\left(1 - \frac{2b}{a\sin^2\gamma}\right)^2 \text{ nearly.}$$

(90) If in the equation

$$\tan\theta = \frac{1}{\cot a_1 + \cot a_2} + \frac{1}{\cot a_3 + \cot a_4}$$

the angles $a_1$, $a_2$, $a_3$, $a_4$ are all nearly equal, then one value of $\theta$ is very nearly $\frac{1}{4}(a_1+a_2+a_3+a_4)$.

(91) $\theta$ differs from $\tan\theta$ by less than $\frac{1}{3}\tan^3\theta$, $\theta$ being less than $\dfrac{\pi}{4}$.

(92) $$\frac{\tan^2\theta-\sinh^2\alpha}{\tan^2\theta-\sinh^2\beta}=\frac{\coth^2\alpha\sin^2\theta-1}{\coth^2\beta\sin^2\theta-1}.$$

(93) $$\sin(\alpha-2n\beta)+\sin\{\alpha-2(n-1)\beta\}+\sin\{\alpha-(2n-2)\beta\}+\ldots$$
$$+\sin\{\alpha+2n\beta\}=\sin\alpha\operatorname{cosec}\beta\sin(2n+1)\beta.$$

(94) $$\tan\alpha\tan(\alpha+\beta)+\tan(\alpha+\beta)\tan(\alpha+2\beta)+\ldots$$
$$+\tan\{\alpha+(n-1)\beta\}\tan(\alpha+n\beta)=\frac{\tan(\alpha+n\beta)-\tan\alpha}{\tan\beta}.$$

(95) $$\sec\alpha\sec(\alpha+\beta)+\sec(\alpha+\beta)\sec(\alpha+2\beta)$$
$$+\sec(\alpha+2\beta)\sec(\alpha+3\beta)$$
$$+\ldots\text{ to }n\text{ terms}$$
$$=\operatorname{cosec}\beta\{\tan(\alpha+n\beta)-\tan\alpha\}.$$

(96) Sum the series
$$\tan^{-1}\frac{4}{1+3.4}+\tan^{-1}\frac{6}{1+8.9}+\tan^{-1}\frac{8}{1+15.16}.$$

(97) $$\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{13}+\tan^{-1}\tfrac{1}{21}+\ldots\text{ to }n\text{ terms}$$
$$=\tan^{-1}\frac{n}{n+2}.$$

(98) Sum to $n$ terms the series
$$1.\tan\theta.\sec2\theta+\sec\theta.\tan2\theta\sec3\theta+\sec2\theta\tan3\theta.\sec4\theta+\ldots.$$

(99) $$(1+\sec2\theta)(1+\sec4\theta)(1+\sec8\theta)\ldots(1+\sec2^n\theta)$$
$$=\frac{\tan2^n\theta}{\tan\theta}.$$

(100) $$(2\cos\theta-1)(2\cos2\theta-1)\ldots(2\cos2^{n-1}\theta-1)=\frac{2\cos2^n\theta+1}{2\cos\theta+1}.$$

(101) Sum the series
$$\frac{\sin\theta}{2\cos\theta+1}+\frac{3\sin3\theta}{2\cos3\theta+1}+\frac{3^2\sin3^2\theta}{2\cos3^2\theta+1}\ldots\text{ to }n\text{ terms.}$$

(102) The $n^{\text{th}}$ convergent of

$$\frac{1}{2\tan a+}\ \frac{1}{2\tan a+}\ \frac{1}{2\tan a+}\ \text{etc., is}$$

$$\frac{(\tan a+\sec a)^n-(\tan a-\sec a)^n}{(\tan a+\sec a)^{n+1}-(\tan a-\sec a)^{n+1}}.$$

(103) The sum of $n$ terms of the series

$$1+\frac{\cos a}{\cos a}+\frac{\cos 2a}{\cos^2 a}+\ldots=0, \text{ if } na=\pi.$$

(104) Sum the series

$$\log(1-2\cos\theta)+\log(1-2\cos 2\theta)+\log(1-2\cos 2^2\theta)+\text{etc.}$$

to $n$ terms.

(105) $$\frac{3\sin x-\sin 3x}{\cos 3x}+\frac{3\sin 3x-\sin 3^2x}{3\cos 3^2x}+\frac{3\sin 3^2x-\sin 3^3x}{3^2\cos 3^3x}+\ldots$$

to $n$ terms $=\frac{3}{2}\left(\frac{\tan 3^n x}{3^n}-\tan x\right)$.

(106) The roots of the equation

$$4x^3+8x^2-19x+8=0$$

are $\sin^2\frac{1}{7}\pi,\ \sin^2\frac{2}{7}\pi,\ \sin^2\frac{5}{7}\pi.$

(107) Solve the equation $x^3+x^2-2x-1=0$.

[Result. $2\cos\frac{2}{7}\pi,\ 2\cos\frac{4}{7}\pi,\ 2\cos\frac{6}{7}\pi$.]

(108) $$(x-\cos\tfrac{2}{5}\pi)\,(x-2\cos\tfrac{4}{5}\pi)\,(x-2\cos\tfrac{6}{5}\pi)\,(x-2\cos\tfrac{8}{5}\pi)$$
$$=x^4+2x^3-x^2-2x+1.$$

(109) The sum to $n$ terms of the series

$$1^2\cos 2a+2^2\cos 4a+3^2\cos 6a+\ldots\text{to } n \text{ terms}$$
$$=\frac{n^2}{2}\frac{\sin(2n+1)a}{\sin a}+\frac{n}{2}\frac{\cos 2na}{\sin^2 a}-\frac{1}{4}\frac{\sin 2na.\cos a}{\sin^3 a}.$$

(110) Sum the series

$$1+\frac{\theta}{1}\frac{\sin 2\theta}{\sin^2\theta}+\frac{\theta^2}{1.2}\frac{\sin 3\theta}{\sin^3\theta}+\frac{\theta^3}{1.2.3}\frac{\sin 4\theta}{\sin^4\theta}+\ldots\text{ad inf.}$$

(111) Sum to $n$ terms

(i) $1+e\sin a+e^2\sin 2a$,

(ii) $\tan\frac{\theta}{2}\sec\theta+\tan\frac{\theta}{2^2}\sec\frac{\theta}{2}+\text{etc.}$

(112) Sum to infinity

(i) $n\sin a+\frac{n(n+1)}{1.2}\sin 2a+\frac{n(n+1)(n+2)}{1.2.3}\sin 3a+\ldots$

(113) Prove that

$$\frac{1-h^2}{(1-h)^2\cos^2\frac{1}{2}x+(1+h)^2\sin^2\frac{1}{2}x}=1+2h\cos x+2h^2\cos 2x+\text{etc.}$$

(114) $$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=4\left\{\frac{a-b}{a+b}\sin^2\theta-\frac{1}{2}\left(\frac{a-b}{a+b}\right)^2\sin^2 2\theta+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3\sin^2 3\theta-\text{etc.}\right\}.$$

(115) $1+\frac{x^2}{3!}+\frac{x^6}{6!}+\ldots=\frac{1}{3}\{e^x+2e^{-\frac{1}{2}x}\cos\frac{1}{2}(x\sqrt{3})\}$.

(116) The roots of the equation

$$x^n\sin na-nx^{n-1}\sin(na+\beta)+\frac{n(n-1)}{2!}x^{n-2}\sin(na+2\beta)-\ldots=0$$

are given by $x=\sin(a+\beta-k\phi)\operatorname{cosec}(\theta-k\phi)$ where $k$ has all integral values from 0 to $n-1$ and $n\phi=\pi$.

(117) Find the general value of $\theta$ which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\ldots(\cos n\theta+i\sin n\theta)=1.$$

(118) When $n$ is even and if $n\phi=\pi$,

$$\tan a\tan(a+\phi)\tan(a+2\phi)\ldots\tan\{a+(n-1)\phi\}=(-1)^{\frac{n}{2}}.$$

(119) When $m$ is odd

$$\tan m\phi=\tan\phi\cot\left(\phi+\frac{\pi}{2m}\right)\tan\left(\phi+\frac{2\pi}{2m}\right)\ldots\cot\left(\phi+\frac{m-2}{2m}\pi\right)\tan\left(\phi+\frac{m-1}{2m}\pi\right).$$

(120) $$\tan\theta+\tan\left(\frac{2\pi}{m}+\theta\right)+\tan\left(\frac{4\pi}{m}+\theta\right)+\ldots\ldots$$
$$+\tan\left(2\frac{m-1}{m}\pi+\theta\right)=\frac{m\sin m\theta}{\cos m\theta-S_m(-1)^{\frac{m}{2}}},$$

where $S_m=\frac{1}{2}$ the sum of the $m^{\text{th}}$ powers of the root of equation $z^2-1=0$.

(121) If $\theta=\dfrac{\pi}{2^n+1}$,

$$2^n\cos\theta\cos 2\theta\cos 2^2\theta\ldots\cos 2^{n-1}\theta=1.$$

(122) If $a, b, c, \ldots$ are the roots of the equation

$$x^n-p_1x^{n-1}+p_2x^{n-2}-p_3x^{n-3}+\text{etc.},$$

then $$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\ldots=\tan^{-1}\frac{p_1-p_3+p_5-\cdots}{1-p_2+p_3-\ldots}.$$

(123) Prove that

$$e^{\pi}+e^{-\pi}=2\,(1+2^2)\,\{1+(\tfrac{2}{3})^2\}\,\{1+(\tfrac{2}{5})^2\}\ldots$$

(124) The sum of the products of the reciprocals of the fourth powers of every positive integer is $\dfrac{384\pi^8}{5!\,9!}$.

(125) $$\tan\frac{y}{2}=\frac{2}{\pi-y}-\frac{2}{\pi+y}+\frac{2}{3\pi-y}-\frac{2}{3\pi+y}+\frac{2}{5\pi-y}-\frac{2}{5\pi+y}.$$

(126) $$\cot\frac{y}{2}=\frac{2}{y}-\frac{2}{2\pi-y}+\frac{2}{2\pi+y}-\frac{2}{4\pi-y}+\frac{2}{4\pi+y}.$$

(127) Prove that the coefficients of $\theta^2$ and $\theta^4$ in the expression

$$\left\{\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\right\}^2+\left\{\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\ldots\right\}^2$$

vanish; explaining *à priori* why they do so.

(128) Having given the formula

$$\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\ldots$$

deduce the expression for $\sin\theta$ in factors.

(129) The coefficient of $x^n$ in the expansion of

$$(1+x)\left(1+\frac{x}{2^2}\right)\left(1+\frac{x}{3^2}\right)\ldots \text{ is } \frac{\pi^{2n}}{(2n+1)!}.$$

(130) By putting $a \pm ia$ for $\theta$ in the expression of $\sin\theta$ in factors, prove that

$$\tan^{-1}\frac{2a^2}{\pi^2}+\tan^{-1}\frac{2a^2}{2^2\pi^2}+\tan^{-1}\frac{2a^2}{3^2\pi^2}+\ldots \text{ ad inf.}$$
$$=\tfrac{1}{4}\pi-\tan^{-1}\{\tanh a\cot a\}+n\pi.$$

(131) If a series of points are distributed symmetrically round the circumference of a circle, the sum of the squares of their distances from a point on the circumference is twice that from the centre.

(132) If $A_1$, $A_2$, $A_3$, ... $A_{2n+1}$ are angular points of a regular polygon inscribed in a circle and $O$ any point in the circumference between $A_1$ and $A_{2n+1}$, then the sum of the lengths

$$OA_1+OA_3+OA_5+\ldots+OA_{2n+1}$$
$$=\text{the sum } OA_2+OA_4+OA_6+\ldots+OA_{2n}.$$

(133) If from a point $P$ straight lines $PB_1$, $PB_2$,...$PB_n$ be drawn to the middle points of the sides of a closed polygon $A_1A_2\ldots A_n$, and if the angles $PB_1A_1$, $PB_2A_2$, ... $PB_nA_n$ be denoted by $a_1$, $a_2$, ... $a_n$ respectively, and the triangles $PA_1A_2$, $PA_2A_3$, ... $PA_nA_1$ by $\Delta_1$, $\Delta_2$, ... $\Delta_n$, prove that

$$\Delta_1\cot a_1+\Delta_2\cot a_2+\ldots+\Delta_n\cot a_n=0.$$

# EXAMINATION PAPERS.

## I. Sandhurst—Further. *Nov.* 1882.

1. Name and define the trigonometrical ratios. Prove that

$$\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A.$$

If the cosecant of an angle between $90^0$ and $180^0$ is $\frac{2}{\sqrt{3}}$, what is the secant? And if the cosine of an angle between $540^0$ and $630^0$ is $-\frac{1}{2}$, what is the cosecant?

2. Prove the following identities:—

i. $(\sin 2A)^2 = 2\cos^2 A\,(1 - \cos 2A)$.

ii. $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$.

iii. $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$.

3. In a plane triangle $ABC$ prove that—

i. $\tan A \tan B \tan C = \tan A + \tan B + \tan C$.

ii. $a \sin A + b \sin B + c \sin C = 2\,(a \cos A + \beta \cos B + \gamma \cos C)$,

where $abc$ are the sides and $\alpha\beta\gamma$ the perpendiculars let fall on them from the opposite angles respectively.

4. Prove that the area of a triangle

$$= \tfrac{1}{4} a^2 \sin 2B + \tfrac{1}{4} b^2 \sin 2A\,;$$

and if $R$, $r$ are the radii of the circumscribing and inscribed circles

$$Rr = \frac{abc}{4\,(a+b+c)}.$$

5. Given $\log 1\frac{1}{5} = \cdot 0791812$ and $\log 2\frac{2}{5} = \cdot 3802112$, find the value of $\sqrt[5]{(3\cdot 6)^3} \times \sqrt[4]{\frac{1}{2\cdot 5}} \div \sqrt[3]{8\frac{4}{7\cdot 2}}$, the mantissae for 46929 and 46930 being 6714413 and 6714506.

In a triangle $ABC$, $b=14$, $c=11$, $A=60^\circ$; find the other angles, having given $L\tan 11^\circ 44' 29'' = 9\cdot 31774$.

6. A measured line is drawn from a point on a horizontal plane in a direction at right angles to the line joining that point to the base of a tower standing on the plane. The angles of elevation of the tower from the two ends of the measured line are $30^\circ$ and $18^\circ$. Find the height of the tower in terms of $l$, the length of the measured line.

## II. Cambridge Previous Examination. *Dec.* 1883.

1. Define the cosecant and tangent of an angle.

Shew that $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

Find all the trigonometrical ratios of $30^\circ$.

2. A man wishes to measure the distance between two points $A$ and $B$ between which lies an obstacle. He therefore walks from $A$ to $C$ in a direction at right angles to $AB$ a distance of 50 yards. He now finds that he can walk directly from $C$ to $B$ and that $CB$ makes an angle of $60^\circ$ with $AC$. Find the distance from $A$ to $B$.

3. Prove that $\sin(90^\circ + A) = \cos A$.

4. Find a formula for all angles having the same tangent as $\alpha$.

Solve completely the equation $\tan^2\theta = 1$.

5. Shew that $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

Prove that

$$\sin(\alpha-\beta)\cos 2\beta + \cos(\alpha-\beta)\sin 2\beta = \sin(\beta-\alpha)\cos 2\alpha + \cos(\beta-\alpha)\sin 2\alpha.$$

If $\sin x = \frac{3}{5}$, $\cos y = \frac{8}{17}$, find $\sin(x+y)$.

6. Prove that

(1) $$\sin\tfrac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}.$$

(2) $\cos 3A = 4\cos^3 A - 3\cos A$.

(3) $\cos 4A = \cos^4 A + \sin^4 A - 6\sin^2 A \cos^2 A$.

Determine the sign of the radical in (1) when $A$ lies between $360^0$ and $720^0$.

7. If $a$, $b$, $c$ be the sides of a triangle $ABC$, shew that

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

If $ABC$ be an equilateral triangle each of whose sides is eight inches, and in $BC$ a point $P$ be taken three inches from $B$, shew that $AP$ is seven inches.

8. In any triangle $ABC$ shew that

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Find all the angles of a triangle whose sides are 13 ft., 14 ft., and 15 ft. in length, having given log 2 = ·30103, log 3 = ·4771213, log 7 = ·8450980 and

$L\tan 26^0\,33' = 9{\cdot}6986847$, tabular difference for $1' = 3159$,
$L\tan 29^0\,44' = 9{\cdot}7567587$, tabular difference for $1' = 2933$,
$L\tan 33^0\,41' = 9{\cdot}8237981$, tabular difference for $1' = 2738$.

9. A base line 400 feet in length is measured from the foot of a vertical tower and at the end of this line the angular elevation of the top of the tower is observed to be $26^0\,33'\,54''$; shew that the height of the tower is very nearly 200 feet.

*Refer to question* 8 *for the necessary logarithms.*

## III. Woolwich—Preliminary. *June*, 1882.

1. Prove that the angle subtended at the centre of a circle by an arc equal in length to its radius is an invariable angle.

One angle of a triangle is $45^0$, and the circular measure of another is $1\frac{1}{2}$. Find the third, both in degrees, and in circular measure.

2. Define the secant of an angle, and shew how your definition applies to angles between $180^0$ and $270^0$.

If $\sec A = -2$, what two values between $0^0$ and $360^0$ may $A$ have?

3. Obtain a formula embracing all the angles which have a given tangent.

Determine all the values of $\theta$ which satisfy the equation:

$$\sqrt{3}\tan^2\theta+1=(1+\sqrt{3})\tan\theta.$$

4. Find an expression for $\tan 3A$ in terms of $\tan A$. Shew also that $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$.

5. Prove that $$\sin 18^0=\frac{\sqrt{5}-1}{4};$$

and that $$\sin^2 30^0=\sin 18^0\sin 54^0.$$

Shew that in any circle the chord of an arc of $108^0$ is equal to the sum of the chords of arcs of $36^0$ and $60^0$.

6. Demonstrate the identities—

(1) $\dfrac{(\operatorname{cosec}A+\sec A)^2}{\operatorname{cosec}^2A+\sec^2A}=1+\sin 2A.$

(2) $\sin 3A=4\sin A\sin(60^0+A)\sin(60^0-A).$

(3) $4(\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5})=\pi.$

7. What are the advantages gained by the use of logarithms calculated to the base 10?

If $\log_{10}2=\cdot30103$, find the logarithms of 5, $\frac{1}{125}$, and $4\sqrt{\cdot005}$, to the base 10.

8. Prove that in any triangle—

(1) $2bc\cos A=b^2+c^2-a^2.$

(2) $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}.$

9. If $r_1$ be the radius of a circle touching the side $a$ of a triangle and the other two sides produced, shew that—

$$r_1\cos\frac{A}{2}=a\cos\frac{B}{2}\cos\frac{C}{2}.$$

If $a$ be the side of a regular polygon of $n$ sides, and $R$, $r$, the radii respectively of its circumscribed and inscribed circles, prove that

$$R+r=\tfrac{1}{2}a\cot\frac{\pi}{2n}.$$

10. Two sides of a triangle, which are respectively 250 and 200 yards long, contain an angle of $54^0\ 36'\ 24''$.

Find the two other angles, having given

$$L\cot 27^0\ 18' = 10{\cdot}2872338, \text{ diff. for } 1' = {\cdot}3100;$$
$$L\tan 12^0\ 8'\ 50'' = 9{\cdot}3329292;\ \log 3 = {\cdot}4771213.$$

11. The eye of a soldier in a straight trench of uniform depth is 2 feet above a level plain on which he sees two men standing in the same straight line as the trench; the parts of their bodies above the level of his eye subtending at it the angles $\tan^{-1}{\cdot}00416$ and $\tan^{-1}{\cdot}004$. On walking 200 ft. towards them in the trench he notices that the height of one exactly hides that of the other; and, on approaching 596 feet 8 in. closer still he finds that the portion of the height of the nearer above the level of his eye subtends at it $45^0$. Find the heights of the men.

## IV. Woolwich—Preliminary. *Dec.* 1882.

1. Shew how to express in degrees, minutes, and seconds, an angle whose circular measure is known.

Find, correct to three places of decimals, the radius of a circle in which an arc 15 inches long subtends at the centre an angle containing $71^0\ 36'\ 3{\cdot}6''$. ($\pi = 3{\cdot}1416$.)

2. Define the *sine* of an angle, and prove that

$$\sin A = \sin(180^0 - A) = \sin\{-(180^0 + A)\}.$$

Write down formulæ including all angles which satisfy—

(1) $2\sin A = 1$,

(2) $2\sin^2 A = 1$.

3. Prove that $\cos(A+B) = \cos A\cos B - \sin A\sin B$, and deduce expressions for $\cos 2A$, $\cos 3A$ in terms of $\cos A$.

4. Given $\cos A = {\cdot}28$, determine the value of $\tan \frac{1}{2}A$, and explain fully the reason of the ambiguity which presents itself in your result.

5. Prove that

(1) $\tan\theta+\cot\theta=\sqrt{\{\sec^2\theta+\operatorname{cosec}^2\theta\}}$.

(2) $\sec\theta-\tan\theta=\tan(\frac{1}{4}\pi-\frac{1}{2}\theta)$.

(3) $\cos 20^0+\cos 100^0+\cos 140^0=0$.

(4) $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

6. State and prove the rules by means of which you can determine by inspection the integral part of the logarithm of any given number.

Given $\log 4{\cdot}96=\cdot6954817$, $\log 4{\cdot}9601=\cdot6954904$, find the logarithms of 496010, ·000496, and 49600·25.

7. Shew that in *any* plane triangle

$$a=b\cos C+c\cos B.$$

If $c=\sqrt{2}$, $A=117^0$, $B=45^0$, find all the other parts of the triangle.

8. Find the greatest angle of the triangle whose sides are 50, 60, 70 respectively, having given

$$\log 6=\cdot7781513,\ L\cos 39^0\,14'=9{\cdot}8890644,\ \text{diff. } 1'=1032.$$

9. Express the area of a triangle in terms of one side and the two angles adjacent to it.

Two angles of a triangular field are $22\frac{1}{2}^0$ and $45^0$ respectively, and the length of the side opposite to the latter is a furlong. Shew that the field contains exactly two acres and a half.

10. Find an expression for the diameter of the circle which touches one side of a triangle and the other sides produced.

If $d_1$, $d_2$, $d_3$ be the diameters of the three escribed circles of a triangle, shew that $d_1d_2+d_2d_3+d_3d_1=(a+b+c)^2$.

11. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at $45^0$ to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the former angles of inclination are changed to $15^0$ and $75^0$ respectively. Find the distance between the boats, and the perpendicular distance of each from the sea-wall.

**V.** MATHEMATICAL TRIPOS. 'The three days.' *Jan.* 1881.

1. Explain, and state the several advantages of, the chief systems of angular measurement in use.

Prove that the circumferences of circles vary as their radii; and mention the approximations to their constant ratio which are practically employed.

Shew that there are eleven pairs of regular polygons which satisfy the condition that the measure of an angle of one in degrees is equal to the measure of an angle of the other in grades: and find the number of sides in each.

2. Define the sine of an angle; and find the value of the sines of angles of $135^0$, $240^0$, $292\frac{1}{2}^0$, $432^0$.

Shew that

$$\sin^2 10^0 + \cos^2 20^0 - \sin 10^0 \cos 20^0 = \sin^2 10^0 + \cos^2 40^0 + \sin 10^0 \cos 40^0 = \tfrac{3}{4}.$$

3. Prove geometrically that

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y).$$

Solve the equation

$$\cos x + \sin 3x + \cos 5x + \sin 7x + \ldots + \sin(4n-1)x = \tfrac{1}{4}(\sec x + \operatorname{cosec} x).$$

4. Find an expression for $\cos(x_1+x_2+x_3)$ in terms of sines and cosines of $x_1$, $x_2$, $x_3$.

State the corresponding theorem for the case of $n$ angles $x_1, x_2, \ldots x_n$.

If $\cos(y-z) + \cos(z-x) + \cos(x-y) = -\frac{3}{2}$, shew that

$$\cos^3(x+\theta) + \cos^3(y+\theta) + \cos^3(z+\theta) - 3\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)$$

vanishes whatever be the value of $\theta$.

5. Shew how to solve a triangle having given the three sides: proving from the formulæ obtained that there cannot be more than one triangle, though there may be none, with the given parts.

The perpendiculars from the angular points of an acute-angled triangle $ABC$ on the opposite sides meet in $P$: and $PA$, $PB$, $PC$ are taken for the sides of a new triangle. Find the condition that this should be possible: and if it is, and the angles of the new triangle are $\alpha, \beta, \gamma$, shew that

$$1 + \frac{\cos\alpha}{\cos A} + \frac{\cos\beta}{\cos B} + \frac{\cos\gamma}{\cos C} = \tfrac{1}{2} \sec A \sec B \sec C.$$

6. Find the radii of the inscribed, the circumscribed, and the nine-point circles of a given triangle.

If $O$ be the centre of the first, $O'$ of the second, and $P$ the centre of perpendiculars, shew that the area of the triangle $OO'P$ is

$$-2R^2 \sin \tfrac{1}{2}(B-C) \sin \tfrac{1}{2}(C-A) \sin \tfrac{1}{2}(A-B),$$

where $R$ is the radius of the circle circumscribing $ABC$.

## VI. Mathematical Tripos, Part I. *June*, 1882.

1. Explain the different methods of measuring angles.

Find the number of degrees in each angle of a regular polygon of $n$ sides (1) when it is convex, (2) when its periphery surrounds the inscribed circle $m$ times.

Find correct to ·01 of an inch the length of the periphery of a decagon which surrounds an inscribed circle of a foot radius three times.

2. Prove geometrically the formula

$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta).$$

Prove that

$$2\cos(\alpha-\beta)\cos(\theta+\alpha)\cos(\theta+\beta) + 2\cos(\beta-\gamma)\cos(\theta+\beta)\cos(\theta+\gamma) + \cos(\gamma-\alpha)\cos(\theta+\gamma)\cos(\theta+\alpha) - \cos 2(\theta+\alpha) - \cos 2(\theta+\beta) - \cos 2(\theta+\gamma) - 1$$

is independent of $\theta$, and exhibit its value as a product of cosines.

3. Prove geometrically the formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}.$$

Prove that if $\alpha$, $\beta$, $\gamma$, $\delta$ be four solutions of the equation

$$\tan(\theta + \tfrac{1}{4}\pi) = 3\tan 3\theta,$$

no two of which have equal tangents, then

$$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = 0,$$

and

$$\tan 2\alpha + \tan 2\beta + \tan 2\gamma + \tan 2\delta = \tfrac{4}{3}.$$

4. Prove that in general the change in the cosine of an angle is approximately proportional to the change in the angle.

Prove that if in measuring the three sides of a triangle small errors $x$, $y$ be made in two of them $a$, $b$, then the error in the angle $C$ will be

$$-\left(\frac{x}{a}\cot B+\frac{y}{b}\cot A\right),$$

and find the errors in the other angles.

5. Prove that in any triangle $a\cos B+b\cos A=c$, and deduce the formula $c^2=a^2+b^2-2ab\cos C$.

Prove that if $O$ be the centre of the circumscribing circle of the triangle $ABC$, the sides of the triangle formed by the centres of the three circles $BOC$, $COA$, $AOB$ will be proportional to

$$\sin 2A : \sin 2B : \sin 2C.$$

Find the angles of the new triangle correct to one second when the sides of the triangle $ABC$ are in the ratio $4 : 5 : 7$.

6. Find the radius of the inscribed circle of a triangle in terms of one side and the angles.

Prove that if $P$ be a point from which tangents to the three escribed circles of a triangle $ABC$ are equal, the distance of $P$ from the side $BC$ will be

$$\tfrac{1}{2}(b+c)\sec\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

## VII. Oxford and Cambridge Schools Examination. *Eton*, 1882.

1. Prove that the cosine and sine of an angle have their signs changed, but their magnitudes unaltered, if the angle be increased by two right angles. Investigate a general formula for all angles whose tangent is equal to $\tan A$.

2. Find the cosine and tangent of $45^0$ and $60^0$. Apply Euclid VI. 3 to find $\tan 15^0$.

3. Prove that $\sin(A-B)=\sin A\cos B-\cos A\sin B$, and that

$$\sin A=\frac{\sin 2A\cos A}{1+\cos 2A}.$$

4. Express $\cos\tfrac{1}{2}(\beta+\gamma-\alpha)\cos\tfrac{1}{2}(\gamma+\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta-\gamma)$, as the sum of cosines of separate angles.

5. Express $\sin \frac{1}{2}A$ in terms of $\sin A$; and prove, *à priori*, that to any given value of $\sin A$, four values of $\sin \frac{1}{2}A$ must correspond.

Having given $\sin 18^0 = \frac{1}{4}(\sqrt{5}-1)$, find $\cos 81^0$.

6. Shew how to find the height of an inaccessible object by observations of its angles of elevation, taken at two points on a straight line through its base.

I stand on a hill on one side of a lake, and observe the angle of elevation ($\alpha$) of the summit of a mountain across the lake, and also its angle of depression ($\beta$) as seen by reflection in the lake. If $h$ be the known height of the mountain, shew that its distance is

$$\frac{2h \cos \alpha \cos \beta}{\sin (\alpha + \beta)},$$

it being given that the ray of light from the top of the mountain makes the same angle with the vertical after reflection from the lake as it did before reflection.

7. Express the sine of half an angle of a triangle in terms of the sides.

Prove that, in any triangle,

$$2 (\cos \tfrac{1}{4}A - \sin \tfrac{1}{4}A)^2 \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$= (\cos \tfrac{1}{2}C + \cos \tfrac{1}{2}A - \cos \tfrac{1}{2}B)(\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B - \cos \tfrac{1}{2}C).$$

8. Find the radius of a circle described about a triangle.

If the radius of this circle be equal to the least side of the triangle, what is the magnitude of the least angle?

9. Sum the series—

(1) $1 + \epsilon^{-mx} \cos nx + \epsilon^{-2mx} \cos 2nx + \ldots\ldots$ *ad infinitum.*

(2) $\cos (\alpha+\beta) + \cos (\alpha+3\beta) + \ldots\ldots + \cos (\alpha + \overline{2n+1}\beta)$.

## VIII. Oxford and Cambridge Schools Examination. *Eton*, 1883.

1. Given $\pi = 3{\cdot}1416$, find the number of degrees in the unit of circular measure of angles.

2. If $\sin \theta = \dfrac{2mn}{m^2+n^2}$, find $\tan \theta$, $\cos 2\theta$.

3. Prove that all angles included in the formula $2n\pi \pm \alpha$ have the same cosine as $\alpha$.

Solve the equation $\cos\theta + \sqrt{3}\sin\theta = 2$.

4. Prove the equivalents:

(1) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$;

(2) $\cos\theta - \cos 3\theta = (\sin 3\theta - \sin\theta)\tan 2\theta$.

5. $ABC$ is a triangle right-angled at $A$; $BD$ meets $AC$ in $D$: find $AD$ in terms of $CD$ and the angles $ABC$, $ABD$.

6. Shew that in any triangle

(1) $\cos\frac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$;

(2) $\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{1}{2}A$;

(3) $(b-c)\cot\frac{1}{2}A + (c-a)\cot\frac{1}{2}B + (a-b)\cot\frac{1}{2}C = 0$.

7. Find an expression for the radius of the circumscribed circle of any triangle in terms of the sides.

The bisector of the angle $A$ meets the side $BC$ in $D$ and the circumscribed circle in $E$: shew that $DE = \frac{a^2 \sec\frac{1}{2}A}{2(b+c)}$.

8. If the ratio of two sides of a triangle is $2+\sqrt{3}$: and the included angle is $60^0$, find the other angles.

9. Shew that $\cos\frac{\theta}{n} + \sqrt{(-1)}\sin\frac{\theta}{n}$ is one of the values of

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{1}{n}},$$

$n$ being a positive integer. What are the other values?

10. Sum the series:

(1) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots$ to $n$ terms;

(2) $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots\ldots$ to infinity.

## IX. Christ's Church, Oxford. Entrance Scholarships. 1883.

1. Prove geometrically that $\theta > \sin\theta > \theta - \frac{\theta^3}{4}$; $\theta$ being less than a right angle.

2. Prove the identities

(i) $\cot(A+B) \equiv \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$, geometrically.

(ii) $(\cos A + \sin A)(\cos 2A + \sin 2A)(\cos A - \sin 3A) \equiv \cos 2A \cos 4A$.

(iii) $2^6(\cos^8 A + \sin^8 A) \equiv \cos 8A + 28\cos 4A + 35$.

(iv) $2\cos(n\cos^{-1}x) = (x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n$.

3. Shew that

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \tfrac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \tfrac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right\}.$$

Having given that $\log_e 3 = 1{\cdot}0986$, find the value of $\log_{10} 3$.

4. Eliminate $\alpha$, $\beta$ from the equations

$$x = (a\sin^2\alpha + b\cos^2\alpha)\cos^2\beta + c\sin^2\beta, \qquad y = a\cos^2\alpha + b\sin^2\alpha,$$
$$z = (b-a)\sin\alpha\cos\alpha\cos\beta.$$

5. If circles can be both described about, and inscribed in a quadrilateral, whose sides are $a$, $b$, $c$, $d$, and the angle between the diagonals $\theta$, then

$$\theta = \cos^{-1}\frac{ac \sim bd}{ac + bd}.$$

6. Solve a triangle, having given the base $a$, altitude $h$, and the difference of the angles of the base $\alpha$.

Account for the two values obtained for the vertical angle, and shew which of them is possible.

7. Shew that in a plane triangle

(i) $\Sigma(b-c)(s-a)\cos A = 0$.

(ii) $4R = r_a + r_b + r_c - r$.

(iii) $\dfrac{C}{2} = \dfrac{c}{a+b}\sin B + \tfrac{1}{2}\left(\dfrac{c}{a+b}\right)^2 \sin 2B + \tfrac{1}{3}\left(\dfrac{c}{a+b}\right)^3 \sin 3B + \ldots$

8. The triangle $A'B'C'$ circumscribes the escribed circles of the plane triangle $ABC$; shew that

$$\frac{B'C'}{a\cos A}=\frac{C'A'}{b\cos B}=\frac{A'B'}{c\cos C}.$$

9. If $K$ be the centre of the nine-point circle of the triangle $ABC$, then $4AK^2=R^2+b^2+c^2-a^2$, where $R$ is the radius of the circumscribing circle.

10. If $\cos(\theta+\phi\sqrt{-1})=\cos\alpha+\sqrt{-1}\sin\alpha$, and $\alpha$, $\theta$, $\phi$ are real, prove that $\tan^2\theta-\tan^2\alpha=\sin^2\theta\sec^2\alpha$, and find a relation between $\theta$ and $\phi$.

11. Sum to infinity the series—

(i) $\cos\alpha\tan\phi-\frac{1}{3}\cos 3\alpha\tan^3\phi+\frac{1}{5}\cos 5\alpha\tan^5\phi\ldots$

(ii) $(1-3^{-\frac{1}{2}})-\frac{1}{3}(1-3^{-\frac{3}{2}})+\frac{1}{5}(1-3^{-\frac{5}{2}})\ldots$

(iii) $\dfrac{1}{\pi^2-x^2}+\dfrac{1}{4\pi^2-x^2}+\dfrac{1}{9\pi^2-x^2}+\ldots$

## X. Christ's College, Cambridge. Entrance Scholarship. 1878.

1. Find the general expression for all angles which have a given tangent or cotangent.

Solve the equation $\sec^2\theta+3\operatorname{cosec}^2\theta=8$.

2. Prove *geometrically* the formulæ:

(1) $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

(2) $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

Shew that

$$\operatorname{cosec}A+\operatorname{cosec}(A+\tfrac{2}{3}\pi)+\operatorname{cosec}(A+\tfrac{4}{3}\pi)=3\operatorname{cosec}3A.$$

3. If $\theta$ be the circular measure of an angle less than a right angle, prove that $\sin\theta$ is less than $\theta$, but greater than $\theta-\frac{1}{4}\theta^3$.

4. Prove that if $\alpha$, $\beta$, $\gamma$ are any three plane angles

$$(\cos\alpha+\cos\beta+\cos\gamma)\{\cos 2\alpha+\cos 2\beta+\cos 2\gamma-\cos(\beta+\gamma)-\cos(\gamma+\alpha)-\cos(\alpha+\beta)\}-(\sin\alpha+\sin\beta+\sin\gamma)\{\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin(\beta+\gamma)-\sin(\gamma+\alpha)-\sin(\alpha+\beta)\}=\cos 3\alpha+\cos 3\beta+\cos 3\gamma-3\cos(\alpha+\beta+\gamma).$$

5. Shew that $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$,

where $R$ is the radius of the circumscribing circle and $r$ of the inscribed circle of the triangle $ABC$.

If $\Delta$, $\Delta'$ be the areas of the two triangles, in the *ambiguous* case (given $A$, $a$, $b$), prove that the continued product of the inscribed and escribed radii to the side $b$ is equal to $\Delta\Delta'$.

6. State De Moivre's theorem, and prove that there are $n$ values and no more for the expression

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{1}{n}}.$$

Write down the fifth roots of $(-1)$.

7. Prove that

$$\sin\theta = n \sin\frac{\theta}{n}\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{\pi}{n}}\right)\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2\frac{2\pi}{n}}\right)\ldots\ldots$$

and deduce the expression for $\sin\theta$ in factors.

Shew that the sum of the series

$$\frac{1}{3^4} + \frac{3}{5^4} + \frac{6}{7^4} + \frac{10}{9^4} + \ldots = \frac{\pi^2}{64}\left(1 - \frac{\pi^2}{12}\right).$$

## XI. St John's College, Cambridge. *June Exam.*, 1879.

1. Explain the method of measuring angles by degrees, minutes, &c.

The numerical measures of the angles $A$, $B$, $C$ of a triangle when referred to units $l^0$, $m^0$, $n^0$, respectively, are in arithmetical progression, and when referred to units $p^0$, $q^0$, $r^0$ respectively, they are in geometrical progression. Find $A$, $B$, $C$.

2. Define the sine and cosine of an angle, and prove that

$$\sin^2 A + \cos^2 A = 1.$$

If $\cos^2 A + \cos B = 1 = \sin^2 A + \sin B$, find $A$ and $B$.

3. Prove geometrically that

(1) $\sin 2A = 2 \sin A \cos A$,

(2) $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$.

If $\theta = \dfrac{\pi}{2^n+1}$, prove that

$$2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = 1.$$

4. Prove that

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B).$$

Find $\theta$ and $\phi$ from the equations

$$\cos\alpha \{\cos\alpha + \cos(\alpha+\theta)\} = \cos\beta \{\cos\beta + \cos(\beta+\phi)\},$$
$$\cos\alpha \{\sin\alpha + \sin(\alpha+\theta)\} = \cos\beta \{\sin\beta + \sin(\beta+\phi)\}.$$

5. If $\theta$ be the circular measure of an angle less than a right angle, prove that $\sin\theta$, $\theta$, and $\tan\theta$, are in ascending order of magnitude.

If the unit of measurement be a right angle, find the limit of $\dfrac{\tan\theta - \sin\theta}{\theta^3}$ as $\theta$ is indefinitely diminished.

6. Expand $\log_a(1+x)$ in a series of powers of $x$.

Prove that

$$2(\cos A + \tfrac{1}{3}\cos^3 A + \tfrac{1}{5}\cos^5 A \ldots\ldots) = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A + \tfrac{1}{3}(\cos^4 \tfrac{1}{2}A - \sin^4 \tfrac{1}{2}A) + \tfrac{1}{5}(\cos^6 \tfrac{1}{2}A - \sin^6 \tfrac{1}{2}A) + \ldots\ldots$$

7. In any triangle the sides are proportional to the sines of the angles opposite to them.

Through the angular point $C$ of a triangle $ABC$ is drawn any line $CMN$ on which are dropped perpendiculars $AM$, $BN$. Prove that

$$MN = AM \cot B \sim BN \cot A.$$

8. Express the sine and cosine of half the angle of a triangle in terms of the sides.

If $ABC$, $A'B'C'$ be two triangles, such that

$$\frac{1+\cos A}{a'} = \frac{1+\cos B}{b'} = \frac{1+\cos C}{c'},$$

prove that $\tan \tfrac{1}{2}A \tan \tfrac{1}{2}A' = \tan \tfrac{1}{2}B \tan \tfrac{1}{2}B' = \tan \tfrac{1}{2}C \tan \tfrac{1}{2}C'$.

9. Give the formulæ for the solution of a triangle in which one angle and the containing sides are given.

If $C = 44^0$, $a = 43$ ft., $b = 11$ ft., find $A$ and $B$.

Having given

$\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, $L \tan 22^0 = 9 \cdot 6064096$,
$L \tan 34^0\, 17' = 9 \cdot 8336109$, $L \tan 34^0\, 19' = 9 \cdot 8338823$.

10. Enunciate and prove De Moivre's theorem.

If $n$ be equal to $3m \pm 1$, prove that

$$(1 \mp n) - \left\{ \frac{n(n-1)}{\lfloor 2} \mp \frac{n(n-1)(n-2)}{\lfloor 3} \right\} 3 + \left\{ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} \mp \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5} \right\} 3^2 - \&c. = 0.$$

11. Find the sum of the following series, each to $n$ terms:

(1) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots\ldots$

(2) $\tan^{-1} 2 + \tan^{-1} \dfrac{4}{1+3\,.\,4} + \tan^{-1} \dfrac{6}{1+8\,.\,9} + \tan^{-1} \dfrac{8}{1+15\,.\,16} + \ldots\ldots$

12. Resolve $x^{2n} - 2x^n \cos\theta + 1$ into factors.

If $n$ be an even integer, prove that

$$\sin^2 \frac{\theta}{2} = 2^{n-2} (-1)^{\frac{n}{2}} \cos\frac{\theta}{n} \cos\frac{2\pi+\theta}{n} \cos\frac{4\pi+\theta}{n} \ldots\ldots \cos\frac{(2n-2)\,\pi+\theta}{n}.$$

## XII. St John's College, Cambridge. Minor Scholarship, 1881.

1. Shew that in the expression for $\tan\dfrac{A}{2}$ in terms of $\tan A$ we should *à priori* expect a double result. Find $\tan 112^0\,.\,30'$.

2. A triangle is such that the product of two sides is equal to the square on half the base: prove that the difference of the sides varies as the distance from the vertex to the middle point of the base.

3. (i) If $x$, $y$, $z$ be any angles, prove that

$$\sin\tfrac{1}{2}(x-y-z)\sin\tfrac{1}{2}(y-z) + \sin\tfrac{1}{2}(x+y-z)\sin\tfrac{1}{2}(y+z) = \sin\tfrac{1}{2}x \sin y.$$

(ii) Also if $A$, $B$, $B$ be the angles of an isosceles triangle,

$$2\sin^2(A-B)(2-\cos A) = (\sin^2 A + 2\sin^2 B)(1 - 8\cos A \cos^2 B).$$

4. (i) Eliminate $\theta$ from

$$\left.\begin{aligned} 2\cos^2\theta + \lambda\sec\theta &= 3 \\ 2\sin^2\theta + \mu\operatorname{cosec}\theta &= 3 \end{aligned}\right\}.$$

(ii) If
$$\left.\begin{aligned} x^2\cos\alpha\cos\beta + x(\sin\alpha+\sin\beta)+1 &= 0 \\ \text{and} \quad x^2\cos\beta\cos\gamma + x(\sin\beta+\sin\gamma)+1 &= 0 \end{aligned}\right\},$$

prove that $x^2\cos\gamma\cos\alpha + x(\sin\gamma+\sin\alpha)+1=0.$

5. Prove that the distance between the centre of the inscribed circle and the intersection of perpendiculars from the angular points on the opposite sides of a triangle is

$$2R\{\operatorname{vers}A \operatorname{vers}B \operatorname{vers}C - \cos A\cos B\cos C\}^{\frac{1}{2}},$$

where $R$ is the radius of the circumscribed circle.

6. Prove that $\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{2}{q}}$ admits of no more than $q$ values.

Find the continued product of the 4 values of

$$\{\cos\tfrac{1}{3}\pi + \sqrt{(-1)}\sin\tfrac{1}{3}\pi\}^{\frac{3}{4}}.$$

## XIII. Clare, Caius, and King's Colleges. *June Exam.*, 1880.

1. Draw a curve representing the change in sign and magnitude of $\tan 2\theta$ while $\theta$ changes from 0 to $\pi$.

Do the same for $\tan 2\theta - 2\tan\theta$.

ii. Prove geometrically

(1) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$,

(2) $\cos 3A = 4\cos^3 A - 3\cos A$.

If $3(1+\tan^2 A\tan^2 B)+8\tan A\tan B = \tan^2 A + \tan^2 B$, $A$ and $B$ differ by some multiple of $\frac{1}{3}\pi$.

3. If $\sin 3A$ be given, and from this value $\tan A$ is to be found, shew *à priori* that six values are to be generally expected.

Prove by help of this, or otherwise, that

$$\tan^2\alpha\{\tan^2(\tfrac{1}{3}\pi-\alpha)+\tan^2(\tfrac{1}{3}\pi+\alpha)\}+\tan^2(\tfrac{1}{3}\pi-\alpha)\tan^2(\tfrac{1}{3}\pi+\alpha) = 6\sec^2 3\alpha + 3.$$

4. If $\theta+\phi+\psi=0$, prove that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\psi=-\frac{\sin\theta+\sin\phi+\sin\psi}{1+\cos\theta+\cos\phi+\cos\psi} \quad \ldots\ldots (1).$$

Find $\cos x$ from the equation

$$\{4\cos(x+\alpha)-1\}\{4\cos(x-\alpha)-1\}=5(2\cos 2\alpha-1) \quad \ldots\ldots(2).$$

Eliminate $\alpha$ from the equations

$$\frac{\sin\theta}{\beta^2-1}=\frac{\cos\theta}{2\beta\sin 2\alpha}=\frac{1}{1+2\beta\cos 2\alpha+\beta^2},$$

shewing that $$\beta=\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta) \quad \ldots\ldots (3).$$

5. State the principle of proportional parts in the use of tables of functions. What is meant by saying that the differences are (1) insensible, (2) irregular?

*Prove* that they are both insensible and irregular in the case of the logarithmic sine when the angle approaches $\frac{1}{2}\pi$.

Determine a limit to the error which can be made in finding the logarithm of $N+\frac{a}{100}$ from seven-figure tables from those of $N$ and $N+1$, where $a$ lies between 0 and 100 and $N$ consists of 5 digits.

6. Explain fully the method of solving a triangle, given two sides, the included angle and a table of logarithms.

$ABC$, $AB'C'$ are two triangles having $AB$, $BC$ equal respectively to $AB'$, $B'C'$, and $A$, $C$, $C'$ are collinear. If the angle $BAB'$ is $1''$, find correctly to a tenth of a second the angle between $BC$ and $B'C'$, where $AB=2BC$ and $\angle ABC=60^0$.

7. $ABC$ is a triangle and tangents are drawn to the nine-point and circumscribing circles at the four points where the perpendicular from $A$ on the opposite side $BC$ meets them.

Prove that the four tangents form a parallelogram of area

$$R^2\frac{\cos A\cos B\cos C}{\tan(B\sim C)}.$$

8. Find the limit of the expression $(90-\theta)\tan\theta^0$ as $\theta$ approaches 90.

9. Given that $x^m+\frac{1}{x^m}=2\cos m\theta$ for all values from 0 to $m$, shew that the formula holds when $m+1$ is written for $m$. Deduce the value of $x^r+\frac{1}{x^r}$, in the most general form.

10. Prove that $\sin\theta=\theta-\frac{\theta^3}{\lfloor\underline{3}}+\frac{\theta^5}{\lfloor\underline{5}}\ldots\ldots$

and deduce the exponential value of $\sin\theta$.

Shew that $\sin^{-1}(\operatorname{cosec}\theta)=(2\lambda+1)\,\frac{1}{2}\pi+\sqrt{(-1)}\,l_e\cot\frac{1}{2}(\lambda\pi+\theta)$, where $\lambda$ is any integer positive or negative.

11. Assuming the factorial expressions for $\sin\theta$ and $\cos\theta$, prove that $\tan\theta>\theta$, provided $\theta$ lie between 0 and $\frac{1}{2}\pi$.

By means of the result in question 8, or otherwise, prove that the infinite product

$$\frac{4\,.\,5}{2\,.\,4}\cdot\frac{5\,.\,6}{4\,.\,6}\cdot\frac{6\,.\,7}{6\,.\,8}\cdot\frac{7\,.\,8}{8\,.\,10}\ldots\ldots \text{ is equal to } \frac{16}{3\pi}.$$

12. Sum the series,

$\cos\alpha+\cos(\alpha+\beta)+\ldots$ to $n$ terms .....(1),

$\sin\alpha+3\sin 2\alpha+5\sin 3\alpha+\ldots$ to $n$ terms.....(2).

## XIV. Christ's, Emmanuel, and Sidney Sussex Colleges.

*June Examination*, 1882.

1. Define the cosine and the tangent of an angle.

Trace the changes in sign and magnitude of

$$(1)\quad \tan\theta, \qquad (2)\quad \frac{\sqrt{3}+\tan\theta}{\sqrt{3}-\tan\theta},$$

as $\theta$ varies from $0^0$ to $360^0$.

2. Prove geometrically that

$$\sin A-\sin B=2\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+B).$$

Shew that

$$(1)\quad \cos 2\alpha \cos^2(\beta+\gamma)+\cos 2\beta \cos^2(\gamma+\alpha)+\cos 2\gamma \cos^2(\alpha+\beta)$$
$$=\cos 2\alpha \cos 2\beta \cos 2\gamma+2\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta),$$

$$\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)}{\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha-\gamma)}\cos\alpha+\frac{\sin\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta+\alpha)}{\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\beta-\alpha)}\cos\beta$$
$$+\frac{\sin\frac{1}{2}(\gamma+\alpha)\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\gamma-\beta)}\cos\gamma=\cos(\alpha+\beta+\gamma).$$

3. Find the limit of $\frac{\sin\theta}{\theta}$ when $\theta$ is diminished indefinitely.

In order to ascertain the distance of an inaccessible object $P$, a person measures a length $AB=20$ yards in a convenient direction; at $A$ he observes that the angle $PAB=60^0$, and at $B$ that the angle $PBA=119^0\ 20'$. Find approximately the distance $BP$. To what degree of accuracy is your result correct, supposing (1) that there is no error in the measurement of the angles, (2) that there is an error of $1'$ in the measurement of each angle?

4. In any triangle $ABC$, shew that

$$a=b\cos c+c\cos B,$$

and that
$$a^2=b^2+c^2-2bc\cos A.$$

If $N$ be the foot of the perpendicular from $C$ on $AB$, and the circle on $CN$ as diameter cut $CA$, $CB$ in $P$ and $Q$ respectively, shew that the angle $BPN$ is equal to the angle $AQN$.

5. Express the area of a triangle in terms of its sides.

A straight line $AB$ is divided at $C$ into two parts of lengths $2a$ and $2b$ respectively. On $AC$, $CB$ and $AB$ as diameters semicircles are described so as to be on the same side of $AB$. If $O$ be the centre of the circle which touches each of the three semicircles, shew that its radius

$$=\frac{ab(a+b)}{a^2+ab+b^2},$$

and that its diameter is equal to the altitude of the triangle $AOB$.

6. Shew how to find the height and distance of an inaccessible object on a horizontal plane.

A person wishing to ascertain the height of a tower stations himself in a horizontal plane through the base at a point at which the elevation at the top is $30^0$. On walking a distance $a$ in a certain direction he finds that the elevation of the top is the same as before, and on walking a distance five-thirds of $a$ at right angles to his previous direction, he finds that the elevation of the top is $60^0$. Shew that the height of the tower is $\sqrt{\frac{5}{6}}a$ or $\sqrt{\frac{5}{4}\frac{5}{8}}a$. Explain the two results.

7. In a triangle $ABC$, $I$, $I'$ and $O$ are the centres of the inscribed circle, the escribed circle opposite $A$ and the circumscribing circle respectively, and $R$ is the radius of the latter circle. Shew that

(1) $OI^2 = R^2(1 - 8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)$,

(2) $\tan IOI' = \pm\dfrac{2(\sin B - \sin C)}{2\cos A - 1}$.

8. Explain the meanings of $\sin^{-1}x$ and $\tan^{-1}x$.

How many bounding lines are required to construct all the angles included in the formula

$$\sin^{-1}a + \cos^{-1}b + \tan^{-1}c?$$

Shew that $\sin^{-1}a + \cos^{-1}b = \sin^{-1}(ab + \sqrt{1-a^2}\sqrt{1-b^2})$.

If $xy + yz + zx = 1$, prove that one of the values of

$$\sin^{-1}\frac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)} + \sin^{-1}\frac{(y+z)(1-yz)}{(1+y^2)(1+z^2)} + \sin^{-1}\frac{(z+x)(1-2x)}{(1+z^2)(1+x^2)} = 2\pi.$$

9. Assuming De Moivre's Theorem find the expansions of $\sin n\theta$ and $\cos n\theta$ as homogeneous functions of $\sin\theta$ and $\cos\theta$.

Find the equation whose roots are $\tan^2\frac{\pi}{11}$, $\tan^2\frac{2\pi}{11}$, $\tan^2\frac{3\pi}{11}$, $\tan^2\frac{4\pi}{11}$, and $\tan^2\frac{5\pi}{11}$. Find also the sum of the fourth powers of these tangents.

10. Investigate Gregory's series for the expansion of $\tan^{-1}x$ in powers of $x$.

Expand $\tan^{-1}(x + \cot\alpha)$ in powers of $x$.

11. Prove that

$$\cos n\theta = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{2n}\right)\left(\cos\theta - \cos\frac{3\pi}{2n}\right)\ldots\ldots\left(\cos\theta - \frac{(2n-1)\pi}{2n}\right).$$

Shew also that if $p<n$

$$\frac{\cos^p\theta}{\cos n\theta}=\frac{1}{n}\sum_{r=0}^{r=n-1}(-1)^r\frac{\sin\frac{(2r+1)\pi}{2n}\cos^p\frac{(2r+1)\pi}{2n}}{\cos\theta-\cos\frac{(2r+1)\pi}{2n}}.$$

12. If $A$, $B$, $C$, $D$.....be the angular points of a regular polygon of $n$ sides inscribed in a circle of radius $a$ and centre $O$, shew that

$$PA^2\,.\,PB^2\,.\,PC^2\ldots=r^{2n}-2a^nr^n\cos n\theta+a^{2n},$$

where $OP=r$ and the angle $AOP=\theta$.

Prove also that the sum of the angles that $AP$, $BP$, $CP$, ... make with $OP$ is $\tan^{-1}\dfrac{r^n\sin n\theta}{r^n\cos n\theta-a^n}$.

# ANSWERS TO THE EXAMPLES.

## II.

(3) The student should observe that each of these series is very slowly convergent.

## III.

(2) $\log_e 7 = 1{\cdot}9479$.

## IV.

Examples (1), (2), (3) indicate a method of obtaining the Logarithmic and Exponential Expansions.

(6) This is a form of Euler's Formula

$$\sin\theta = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

when $\theta$ is indefinitely increased.

## VI.

(8) (i) $(-1)^{\frac{n-2}{n}} s_{n-1}$. (ii) $(-1)^{\frac{n-1}{2}} s_n$.

## VII.

(1) $\sqrt{2}\{\cos(-45^0) + i\sin(-45^0)\}$, $2(\cos 30^0 + i\sin 30^0)$, $2(\cos 60^0 + i\sin 60^0)$.

(2) (i) $2(\cos 15^0 + i \sin 15^0)$, $2(\cos 135^0 + i \sin 135^0)$, $2(\cos 255^0 + i \sin 255^0)$.

(ii) $2(\cos 10^0 + i \sin 10^0)$, $2(\cos 130^0 + i \sin 130^0)$, $2(\cos 250^0 + i \sin 250^0)$.

(iii) $\sqrt[5]{2}(\cos 6^0 + i \sin 6^0)$, $\sqrt[5]{2}(\cos 78^0 + i \sin 78^0)$, $\sqrt[5]{2}(\cos 150^0 + i \sin 150^0)$, $\sqrt[5]{2}(\cos 222^0 + i \sin 222^0)$, $\sqrt[5]{2}(\cos 294^0 + i \sin 294^0)$.

(3) (i) $\pm 1$, $\pm\sqrt{(-1)}$.

(ii) 2, $2\{\cos\frac{1}{6}(2r\pi) + i \sin\frac{1}{6}(2r\pi)\}$ putting 1, 2, 3, 4 for $r$ successively.

(iii) 3, $\frac{3}{2}\{-1 + i\sqrt{3}\}$, $\frac{3}{2}\{-1 - i\sqrt{3}\}$.

## VIII.

(1) $1, \cos\frac{1}{6}(2r\pi) + i \sin\frac{1}{6}(2r\pi)$, where $r = 1, 2, 3, 4$.

(2) $(x^2-1)(x^2-\sqrt{2}x+1)(x^2+1)(x^2+\sqrt{2}x+1)$.

(3) $(x-1)\{x^2 - 2x\cos\frac{1}{12}(2r\pi) + 1\}$ six factors putting 1, 2, 3, 4, 5, 6 for $r$.

(4) $(x^3-1)(x^2-x+1)(x^2+x+1)$. (5) $x^2 - 2ax\cos\frac{1}{10}(r\pi) + a^2$.

(6) $\cos\frac{1}{6}(r\pi) + i \sin\frac{1}{6}(r\pi)$, $r$ having each integral value from 0 up to 11.

## IX.

(1) $\pm\frac{1}{\sqrt{2}} \pm \sqrt{(-1)}\frac{1}{\sqrt{2}}$, $(x^2 - x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$.

(2) $(x^2-\sqrt{3}x+1)(x^2+1)(x^2+\sqrt{3}x+1)$.

(3) $x^2 - 2x\cos(1+2r)9^0 + 1$.

(4) Solve the equation $(x^2-\sqrt{3}x+1)(x^2+1)(x^2+\sqrt{3}x+1) = 0$.

(5) $(x+1)\{x^2 - 2x\cos\frac{1}{15}(\pi + 2r\pi) + 1\}$ seven factors in all.

(6) $\cos\frac{\pi+2r\pi}{n} + i \sin\frac{\pi+2r\pi}{n}$, where $r$ may have any integral value.

## X.

(3) With the notation of Ex. (1)

$$\sin(\alpha+\beta+\gamma+\delta+\epsilon)=S_1C_4-S_3C_2+S_5$$
$$\cos(\alpha+\beta+\gamma+\delta+\epsilon)=C_5-S_2C_3+S_4C_1.$$

## XIII.

(9) Put $b=0$ in Ex. 3, p. 26.

(10) Put $\alpha=0$, $\beta=1$ in Ex. 3, p. 26.

(17) $\sqrt[3]{(-1)}=\cos\dfrac{\pi+2r\pi}{3}+i\sin\dfrac{\pi+2r\pi}{3}$; making this substitution $a^{\sqrt[3]{i}}$ is expressed in the form $a^{\alpha+ib}$. Then proceed as in Ex. 3, p. 26.

(18) $$c^i=e^{i\log c}=\cos(\log c)+i\sin(\log c),$$
$$\therefore\ a+ib+c^i=\{a+\cos(\log c)\}+i\{b+\sin(\log c)\}.$$

Then proceed as in Ex. 3, p. 26.

## XIV.

(1) The equation $e^{6i\pi}=e^0$, does not assert that $6i\pi=0$, but that $6i\pi=0+2ir\pi$.

(2) $e^{i(\alpha-\pi)}=e^{i(\alpha+\pi)}$, does not assert that $i(\alpha-\pi)=i(\alpha+\pi)$.

## XVII.

(1) $(x^2-2x\cos 15^0+1)(x^2-2x\cos 105^0+1)(x^2-2\cos 195^0+1)$
$(x^2-2x\cos 285^0+1)=0.$

(2) $(x^2-2x\cos 2^0+1)(x^2-2x\cos 74^0+1)(x^2-2x\cos 146^0+1)$
$\times(x^2-2x\cos 218^0+1)(x^2-2x\cos 290^0+1)=0.$

(3) $x^2-2x\cos\frac{1}{9}(3r\pi+\pi)+1=0$, six quadratics.

(4) $x^2+2x\cos(r\times 72^0+6^0)+1=0$, five quadratics.

(5) $x^2-2xy\cos\dfrac{\alpha+2r\pi}{n}+y^2$, $n$ factors.

## XIX.

(5) $\theta$ is the smaller root of the quadratic

$$\theta^2 - 2\theta + (\cdot 0029\ldots)2\sqrt{2} = 0.$$

(7) $2m^{-2}$. (8) $\dfrac{2n^2 - 2m^2}{p^2}$. (9) $\dfrac{2(m-n)^2}{3mn}$.

(10) $\frac{1}{12}$.

## XXI.

(4) (i) $\dfrac{2n(2n-1)\ldots(n+1)}{n(n-1)\ldots 2.1}$,

(ii) $\dfrac{(2n+1)\,2n(2n-1)\ldots(n+2)}{n.(n-1)\ldots 2.1} 2\cos\theta$.

(iii) $-\dfrac{(4n+2)(4n+1)\ldots(2n+2)}{(2n+1)\,2n(2n-1)\ldots 2.1}$.

## XXIV.

(1) $n$ even, $\{(-1)^{\frac{n}{2}} - \cos n\alpha\}\,2^{-n+1}$; $n$ odd, $2^{-n+1}\cos n\alpha$.

(2) $n$ even, 0; $n$ odd $(-1)^{\frac{n-1}{2}}\dfrac{n}{\cos n\alpha}$.

(3) $n$ even, $(-1)^{\frac{n}{2}}(1-\cos n\alpha)\,2^{-n+1}$; $n$ odd $(-1)^{\frac{n-1}{2}}\,2^{-n+1}\sin n\alpha$.

(4) $n$ even, $\dfrac{n^2}{1-\cos n\alpha}$; $n$ odd $\dfrac{n^2}{\sin^2 n\alpha}$.

(5) (6) (7) $$\tan n\alpha = \frac{n\tan\theta - \dfrac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \ldots}{1 - \dfrac{n(n-1)}{1.2}\tan^2\theta + \text{etc.}}$$

by Art. 40 is an equation of the $n^{\text{th}}$ degree in $\tan\theta$, of which $\tan\alpha$ is one root, and $\tan\left(\alpha + \dfrac{r\pi}{n}\right)$ is another. Hence as in Art. 61.

(5) The sum, $n$ even, $= \dfrac{n^2}{\tan^2 n\alpha} + n(n-1)$;

$n$ odd $= n^2\tan^2 n\alpha + n(n-1)$.

(6) The sum, $n$ even, $= -n\cot n\alpha$; $n$ odd, $= n\tan n\alpha$.

(7) The sum $= n\cot n\alpha$.

## XXV.

(3) Let $u=(\cos\theta)^{\cot\theta}$, then $\log u=\cot\theta\log(\cos\theta)$

$$=(1-\tfrac{1}{2}\theta^2+\ldots)(-\tfrac{1}{2}\theta^2+\ldots)\div(\theta-\tfrac{1}{6}\theta^3+\ldots)$$

the limit of this $=0$ when $\theta=0$, therefore the limit of $u$ is 1.

(4) $-\infty$. (5) Of the third order.

(8) $$2^{-4n+2}\left\{\cos 4n\theta+\frac{4n(4n-1)}{1\cdot2}\cos 4(n-1)\theta\right.$$
$$\left.+\frac{4n(4n-1)(4n-2)(4n-3)}{1\cdot2\cdot3\cdot4}\cos 4(n-2)\theta+\ldots\right\}$$

(10) By Art. 40, or Ex. 12, p. 43, the equation

$$\tan n\alpha=\frac{n\tan\theta-\dfrac{n(n-1)(n-2)}{1\cdot2\cdot3}\tan^3\theta+\ldots}{1-\dfrac{n(n-1)}{1\cdot2}\tan^2\theta+\ldots}$$

has for its roots $\alpha$, $\alpha+\dfrac{\pi}{n}$, $\alpha+\dfrac{2\pi}{n}$, ...

Put $\alpha=0$, $n=11$, and divide by $\tan\theta$, when we have an equation in $\tan\theta$, viz.

$$0=1-15\tan^2\theta+42\tan^4\theta-30\tan^6\theta+5\tan^8\theta-\tfrac{1}{11}\tan^{10}\theta,$$

whose roots are $\pm\tan\frac{1}{11}\pi$, $\pm\tan\frac{2}{11}\pi$, ... $\pm\tan\frac{5}{11}\pi$,
writing $x$ for $\tan^2\theta$, we have the required equation.

## XXVI.

(1) $\dfrac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}$. (2) $\dfrac{\cos n\alpha\,.\,\sin n\alpha}{\sin\alpha}$.

(3) $\dfrac{\sin\frac{1}{2}(3n-1)\alpha\sin\frac{3}{2}n\alpha}{\sin\frac{3}{2}\alpha}$. (4) $\dfrac{\sin(n+1)\alpha\sin n\alpha}{2\sin\alpha}$.

(5) $\frac{1}{2}\left\{n+\dfrac{\cos(n+1)\alpha\sin n\alpha}{\sin\alpha}\right\}$. (6) Use $\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin 3\alpha)$.

(7) Use $\cos^4\alpha=\frac{1}{8}\cos 4\alpha+\frac{1}{2}\cos 2\alpha+\frac{3}{8}$.

(8) $=\frac{1}{2}(\sin 3\alpha+\sin\alpha)+\frac{1}{2}(\sin 5\alpha+\sin\alpha)+\frac{1}{2}(\sin 7\alpha+\sin\alpha)+\ldots$

(9) $=\frac{1}{2}\{\cos\alpha-\cos 3\alpha+\cos\alpha-\cos 5\alpha+\cos\alpha-\cos 7\alpha+\ldots\}$.

(10) Use $\cos^3\alpha=\frac{1}{4}(\cos 3\alpha+3\cos\alpha)$.

(11) Use $\sin^4\alpha=\frac{1}{8}\sin 4\alpha-\frac{1}{2}\sin 2\alpha+\frac{3}{8}$.

(12) $\theta=\dfrac{2r\pi}{n}$, or $\dfrac{2}{n+1}(r\pi+\frac{1}{4}\pi)$ where $r$ is any integer.

(13) 0 and $\frac{3}{8}n$.

(16) $\frac{1}{2}n(n+1)$. (17) $\frac{1}{4}n^2(n+1)^2$. (18) $\frac{1}{3}n(n+1)(n+2)$.

(19) Write $\delta+\pi$ for $\delta$. (20) Write $\delta+\pi$ for $\delta$.

## XXVII.

(1) $\operatorname{cosec}\theta\{\tan(n+1)\theta-\tan\theta\}$.

(2) $\operatorname{cosec}\theta\{\cot\theta-\cot(n+1)\theta\}$.

(3) $\sec\theta\{\tan(n+1)(\theta-\frac{1}{2}\pi)+\cot\theta\}$, this may be proved by putting $\theta-\frac{1}{2}\pi$ for $\theta$ in Ex. (1).

(4) Each term in this series is one-half the corresponding term in Ex. (1).

(5) $\frac{1}{4}\operatorname{cosec}\frac{1}{2}\alpha\{\sec\frac{1}{2}(2n+1)\alpha-\sec\frac{1}{2}\alpha\}$.

(6) $\frac{1}{4}\operatorname{cosec}\frac{1}{2}\alpha\{\operatorname{cosec}\frac{1}{2}\alpha-\operatorname{cosec}\frac{1}{2}(2n+1)\alpha\}$.

(7) $\frac{1}{2}\sin 2\theta-\dfrac{1}{2^{n+1}}\sin 2^{n+1}\theta$.

(8) $\frac{1}{2}\sin 2\theta-(-1)^n\dfrac{1}{2^{n+1}}\sin 2^{n+1}\theta$.

(9) $\frac{1}{4}\sin 2\theta-(-1)^n 2^{n-2}\sin\dfrac{\theta}{2^{n-1}}$.

(10) The result is similar to that in Ex. (5).

(11) $\frac{1}{2}\sec\alpha\{\operatorname{cosec}\alpha-(-1)^n\operatorname{cosec}(2n+1)\alpha\}$.

(12) $\tan^{-1}(n+1)-\tan^{-1}1$.

(13) $\tan^{-1}(n+1)\alpha-\tan^{-1}\alpha$.

(14) $\tan^{-1}(n+1)^2\alpha^2-\tan^{-1}\alpha^2$.

(15) $\cot\theta-2^n\cot 2^n\theta$.

(16) The series reduces to [See Ex. 5, p. 66]

$$2\,(\operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha + \operatorname{cosec} 8\alpha + \ldots \text{ to } n \text{ terms}).$$

(17) $\frac{1}{2}\{\cos 2\theta - \cos 2^{n+1}\theta\}$. (18) $\frac{1}{2}\left\{\cos\frac{\theta}{2^{n-2}} - \cos 4\theta\right\}$.

(19) $\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} - 2^{n-1}\operatorname{cosec}^2 2^{n-1}\theta$. (20) $\frac{1}{2^n}\operatorname{cosec}^2\frac{\theta}{2^n} - \operatorname{cosec}^2\theta$.

(21) $\cot\frac{\alpha}{2^n} - \cot\alpha$.

(22) The series reduces to [Ex. 5, p. 66]

$$\sin\alpha\,(\operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha + \operatorname{cosec} 8\alpha + \ldots \text{ to } n \text{ terms}).$$

(23) $1 - \frac{1}{n+1}$. (24) $2^{2n} - 1$.

## XXVIII.

(1) $\frac{\sin\alpha}{1 - 2x\cos\alpha + x^2}$. (2) $\frac{\cos\alpha - x\cos(\alpha-\beta)}{1 - 2x\cos\beta + x^2}$.

(3) $\frac{\sin\alpha - \cos\alpha\sin(\alpha-\beta)}{1 - 2\cos\alpha\cos\beta + \cos^2\alpha}$. (4) $\frac{\cos\alpha - \sin\alpha\cos(\alpha-\beta)}{1 - \sin 2\alpha + \sin^2\alpha}$.

(5) $e^{\cos\alpha}\sin(\sin\alpha)$. (6) $e^{x\cos\alpha}\sin(x\sin\alpha)$.

(7) $e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\sin\beta)$.

(8) $e^{-\cos\alpha\cos\beta}\sin(\alpha - \cos\alpha\sin\beta)$. (9) $\frac{1}{2}\alpha$.

(10) $-\log(2\sin\frac{1}{2}\alpha)$. (11) $\frac{1}{2}\log(1 + 2\sin\alpha\cos\beta + \sin^2\alpha)$.

(12) $-\log(1 + 2\cos\theta\cos\phi + \cos^2\theta)$.

(13) $\frac{\cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos\{\alpha+(n-1)\beta\}}{1 - 2x\cos\beta + x^2}$.

(14)

$$\frac{x\sin\alpha + x^2\sin(\alpha-\beta) + (-x)^{n+1}\sin(\alpha+n\beta) - (-x)^{n+2}\sin\{\alpha+(n-1)\beta\}}{1 + 2x\cos\beta + x^2}.$$

(15) $2^n\cos^n\frac{1}{2}\alpha\cos\frac{1}{2}n\alpha$.

(16) $r^n \sin(\alpha+n\phi)$, when $r^2=1+2x\cos\alpha+x^2$, and

$$\tan\phi=\frac{x\sin\beta}{1+x\cos\beta}.$$

(17) $$\frac{1-\cos\alpha\cos\beta-\cos^n\alpha\{\cos n\beta-\cos\alpha\cos(n-1)\beta\}}{1-2\cos\alpha\cos\beta+\cos^2\alpha}.$$

(18) $$\frac{\sin\alpha\{1-\sin(\alpha-\beta)\}-\sin^n\alpha\{\sin(\alpha+n\beta)-\sin\alpha\sin(\alpha+\overline{n-1}\beta)\}}{1-2\sin\alpha\cos\beta+\sin^2\alpha}.$$

(19) $$\frac{(n+1)\sin n\alpha-n\sin(n+1)\alpha}{2(1-\cos\alpha)}.$$

(20) $$\frac{n^2\{\cos n\alpha-\cos(n+1)\alpha\}+2n\cos n\alpha}{2(1-\cos\alpha)}-\frac{\sin\alpha\sin n\alpha}{2(1-\cos\alpha)^2}.$$

(21) $\frac{r}{r^q}\sin\frac{p\phi}{q}$, where $r^2=1+2\cos\alpha\cos\beta+\cos^2\alpha$, and

$$\tan\phi=\frac{\cos\alpha\sin\beta}{1+\cos\alpha\cos\beta}.$$

(22) The sum of $x\cos\theta-\frac{1}{3}x^3\cos3\theta+\text{etc.}=\frac{1}{2}\tan^{-1}\frac{2x\cos\theta}{1-x^2}$; $\therefore$ the required sum $=\frac{1}{2}(2n+1)\pi$.

(23) $e^{e^{\cos\alpha}\cos(\sin\alpha)}\cos\{e^{\cos\alpha}\sin(\sin\alpha)\}$.

(24) $\frac{1}{2}\tan^{-1}\frac{\cos y}{\sinh x}$. (25) $\tan^{-1}\frac{e^x\sin x}{1+e^x\cos x}$.

## XXIX.

(2) $\sin\alpha+x\sin2\alpha+x^2\sin3\alpha+\ldots$

(3) $1-x\cos\theta+x^2\cos2\theta-x^3\cos3\theta+\text{etc.}$

(4) $-2\{x\cos\alpha+\frac{1}{2}x^2\cos2\alpha+\frac{1}{3}x^3\cos3\alpha+\ldots\}$.

(5) $\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\ldots$

(6) $1+x\cos\alpha+\frac{x^2\cos2\alpha}{1\cdot2}+\frac{x^3\cos3\alpha}{1\cdot2\cdot3}+\ldots$

(7) $1+rx\cos\left(\tan^{-1}\frac{b}{a}\right)+\frac{r^2x^2}{1\cdot2}\cos2\left(\tan^{-1}\frac{b}{a}\right)+\ldots$

where $r^2=a^2+b^2$.

(8) The coefficient of $x^n$ is $\dfrac{2(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\cos\dfrac{n\pi}{4}\cos\left\{\dfrac{n}{2}\tan^{-1}\dfrac{a^2-b^2}{2ab}\right\}$.

(9) $x\sin\alpha+\dfrac{x^2}{1\,.\,2}\sin 2\alpha+\dfrac{x^3}{1\,.\,2\,.\,3}\sin 3\alpha+\ldots$

(10) $1+x\cos(\alpha+\beta)+\dfrac{x^2}{1\,.\,2}\cos(\alpha+2\beta)+\ldots$

(11) $\sin\alpha+x\sin(\alpha+\beta)+\dfrac{x^2}{1\,.\,2}\sin(\alpha+2\beta)+\dfrac{x^3}{1\,.\,2\,.\,3}\sin(\alpha+3\beta)+\ldots$

(13) Here $\dfrac{e^{i\phi}-e^{-i\phi}}{e^{i\phi}+e^{-i\phi}}=n\,\dfrac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$.

$\therefore\ e^{2i\phi}=e^{2i\theta}\dfrac{1+re^{-2i\theta}}{1+re^{2i\theta}}$ where $r=\dfrac{1-n}{1+n}$.

$\therefore\ 2i\phi+2im\pi=2i\theta+\log(1+re^{-2i\theta})-\log(1+re^{2i\theta})$.

$\therefore\ \phi+m\pi=\theta-r\sin 2\theta+\tfrac{1}{2}r^2\sin 4\theta-\ldots$

(14) $\sec^n\theta=2^n\left\{\cos n\theta-n\cos(n+2)\theta+\dfrac{n(n+1)}{1\,.\,2}\cos(n+4)\theta-\ldots\right\}$

(15) $\cos n\alpha\cos^n\alpha=1-\dfrac{n(n+1)}{1\,.\,2}\tan^2\alpha+\dfrac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}\tan^4\alpha-\ldots$

(16) These series are recurring series

(i) $\dfrac{1-3\cos\theta}{1-6\cos\theta+9}+\dfrac{3(1-2\cos\theta)}{1-4\cos\theta+4}$;

(ii) $\dfrac{1}{3}\left\{\dfrac{x\sin\theta}{1-2x\cos\theta+x^2}+\dfrac{8x\sin\theta}{1-8x\cos\theta+16x^2}\right\}-1$.

## XXX.

(1) Use the second of the identities in Art. 81.

(5) Put $\frac{1}{2}\pi$ for $\theta$ in the expression of $\sin\theta$ in factors. This expression is known as Wallis's expression for $\pi$.

(15) Put $ix$ for $\theta$ in both sides of the identity.

(17) $\dfrac{\sin(\theta+\alpha)}{\sin\alpha} = \cos\theta + \cot\alpha \sin\theta$; use this transformation in Ex. (8), expand $\cos\theta + \cot\alpha \sin\theta$ in ascending powers of $\theta$ by Art. 41 and equate the coefficients of $\theta$ on each side.

(19) This result may be deduced from Ex. (13).

(20) Put $\frac{1}{2}\pi - \alpha$ for $\alpha$ in (19).

## XXXII.

(1) (i) 8·4832462. (ii) 8·0563377. (iii) 8·4834473.

(2) (i) $1^0\,44'\,36''{\cdot}8$. (ii) $55'\,37''{\cdot}4$. (iii) $1^0\,44'\,36''{\cdot}8$.

## XXXIX.

(1) $-1{\cdot}708\,\{1 \pm 2{\cdot}0035\}$. (2) $3{\cdot}9705\,\{1 \pm {\cdot}91032\}$.

## XL.

(1) (i) $2\cos 40^0$, $2\cos 160^0$, $2\cos 80^0$.

(ii) $\sqrt{2}\cos 15^0$, $\sqrt{2}\cos 135^0$, $\sqrt{2}\cos 105^0$.

(iii) $2\cos 40^0 + 1$, $2\cos 160^0 + 1$, $2\cos 80^0 + 1$.

(iv) $2\sqrt{2}\cos 45^0 - 4$, $2\sqrt{2}\cos 165^0 - 4$, $2\sqrt{2}\cos 75^0 - 4$.

(v) $4\cos 10^0 - \sqrt{3}$, $4\cos 130^0 - \sqrt{3}$, $4\cos 110^0 - \sqrt{3}$.

(2) (i) $3\alpha = 1^0\,22'$. (ii) $3\alpha = 65^0\,41'\,53''$. (iii) $3\alpha = 74^0\,55'\,47''$.

(3) (i) Find $\alpha$ and $\beta$ such that $a = \tan\alpha$, $b = \tan\beta$, then

$$a \pm b = \sin(\alpha \pm \beta)\sec\alpha\sec\beta.$$

(ii) $a\cos\theta \pm b\sin\theta = a\cos(\theta \mp \alpha)\sec\alpha$, where $\tan\alpha = \dfrac{b}{a}$.

(iii) $4\sin\frac{1}{2}(B+C)\sin\frac{1}{2}(C+A)\sin\frac{1}{2}(A+B)$.

(iv) $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\alpha)$.

(v) $4R\sin A\sin B\sin C$.

## XLI.

(2) Divide the first equation by $a$, the second by $b$, square both sides and add.

(3) The second equation gives $\cos^2\alpha = \dfrac{y-b}{a-b}$.

(6) $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2 - 3\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right) - \dfrac{2x}{a} = 0.$

(7) $b^2 = c^2 - 2ac\cos 2\phi + a^2.$

(9) $\cos 4\theta = \frac{1}{3}(2m^2-5)$, $\sin(\alpha - 4\theta) = \frac{1}{3}\sin\alpha$, and so on.

## XLIII.

(85) $\frac{1}{16}\pi^3$.

(91) $\sin\theta$ is greater than $\theta - \frac{1}{6}\theta^3$, when $\theta$ is less than $\frac{1}{2}\pi$

$\cos\theta$ is less than $1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$. [Art. 44, 45.]

Therefore $\tan\theta$ is greater than $\dfrac{\theta - \frac{1}{6}\theta^3}{1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4}$.

That is, $\tan\theta$ is greater than $\theta + \frac{1}{3}\theta^3 + \dfrac{\frac{1}{6}\theta^5(1-\frac{1}{9}\theta^2)}{1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4}$; and if $\theta$ is less than $\frac{1}{2}\pi$ the last fraction is positive.

(96) $\tan^{-1}\frac{1}{2} - \tan^{-1}\dfrac{1}{(n+1)(n+2)}$.

(98) $\dfrac{1}{2\sin\theta\cos n\theta\cos(n+1)\theta} - \dfrac{1}{\sin 2\theta}$.

(101) $\frac{1}{4}\{\cot\frac{1}{2}\theta - 3^n\cot\frac{1}{2}(3^n\theta)\}$.

(104) See Ex. 100. (110) $2\cos\theta\, e^{\theta\cot\theta}$.

(111) (i) $1 + \dfrac{e\sin\alpha - e^n\sin n\alpha + e^{n+1}\sin(n-1)\alpha}{1 - 2e\cos\alpha + e^2}$.

(ii) $\tan\theta - \tan\dfrac{\theta}{2^n}$. (112) $\dfrac{-\sin\frac{1}{2}n(\pi-\alpha)}{2^n\sin^n\frac{1}{2}\alpha}$.

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## V.

3. $x = r\pi - \frac{1}{4}\pi$, $4nx = r\pi + \frac{1}{4}\pi$.

## VI.

2. $2\cos(\beta - \gamma)\cos(\gamma - \alpha)\cos(\alpha - \beta)$.

4. $\dfrac{x}{b\sin C} - \dfrac{y\cot C}{b}$, $\dfrac{y}{a\sin C} - \dfrac{x}{a}\cot C$.

VII.

4. $\frac{1}{4}\{\cos\frac{1}{2}(\alpha+\beta+\gamma)+\cos\frac{1}{2}(\beta+\gamma-3\alpha)+\cos\frac{1}{2}(\gamma+\alpha-3\beta)+\cos\frac{1}{2}(\alpha+\beta-3\gamma)\}$.

9. (1) $e^{e^{-mx}\cos(e^{-mx}\sin nx)}$; (2) $\cos\{\alpha+(n+1)\beta\}\sin n\beta\operatorname{cosec}\beta$.

VIII.

2. $\tan\theta=\pm\dfrac{m^2-n^2}{2mn}$, $\cos 2\theta=\dfrac{m^4+n^4-6m^2n^2}{(m^2+n^2)^2}$.

3. $\cos(\theta+\frac{1}{3}\pi)=1$. 5. $AD=\dfrac{CD\sin ABD\cos ABC}{\sin(ABC-ABD)}$.

8. $B-C=90^0$. 10. (2) $\frac{1}{2}\theta-\frac{1}{2}(2r+1)\pi$.

IX.

11. (i) See Ex. XXVIII. (22). (ii) See Art. 32.
(iii) See Ex. XXX. (18).

XI.

1. $\dfrac{A^0}{l}+\dfrac{C^0}{n}=\dfrac{2B^0}{m}$, $\dfrac{A^0C^0}{pq}=\left(\dfrac{B^0}{r}\right)^2$, $A^0+B^0+C^0=180^0$.

2. $\theta=2\beta+2n\pi$, $\phi=2\alpha+2n\pi$. 5. $\frac{1}{10}\pi^3$.

11. (2) See XLIII. (96).

XII.

6. $-1$.

XIII.

12. (2) $\dfrac{(2n+1)\sin n\alpha-(2n-1)\sin(n+1)\alpha-\sin\alpha}{2(1-\cos\alpha)}$.

XIV.

9. See answer to Ex. XXV. (10).

10. $r\pi+\frac{1}{2}\pi-\{x^{-1}-\frac{1}{2}x^{-2}(\sin\alpha)^{-2}\sin 2\alpha+\frac{1}{3}x^{-3}(\sin\alpha)^{-3}\sin 3\alpha-\ldots\}$.

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